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ELEMENTS OF GEOMETRY

46.
916.

THE

ELEMENTS OF GEOMETRY,

SYMBOLICALLY ARRANGED.

PUBLISHED BY COMMAND OF THE LORDS COMMISSIONERS
OF THE ADMIRALTY,

FOR THE USE OF THE BOYS OF THE ROYAL HOSPITAL SCHOOLS,
GREENWICH.

SECOND EDITION.

LONDON:
JOHN MURRAY, ALBEMARLE STREET.

1846.

GEOMETRY is divided into *Plane* and *Solid* Geome
The former is confined to the consideration of the]
perties of Space within the same plane, and the la
to the relations that exist between different Plane
Surfaces, or of the Solids which these describe or
minate. In this book, the points, lines, and surf
are all considered to be in the same plane; and the
of the first books of Euclid has been adhered t
closely in this little work as its symbolical form,
the object for which it is intended, will admit of.

The Learner is earnestly recommended to repeat
tinctly to himself, each step in the demonstratio
the propositions; and moreover to refer to the pr
tions upon which the proofs depend. He wil
acquire a facility of not only *writing down* the p
tions with neatness and accuracy; but also, wher
upon to demonstrate them by *word of mouth*, of
ing himself in clear and intelligible language.

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THE

ELEMENTS OF GEOMETRY.

ABBREVIATIONS.

adj.	signifies	adjacent.	i. e.	signifies	that is.
ax.	„	axiom.	isosc.	„	isosceles.
$\odot^{ce}$	„	circumference.	oppo.	„	opposite.
cr.	„	centre.	post.	„	postulate.
const.	„	construction.	prod.	„	produce. produced.
def.	„	definition.	prop.	„	proposition.
descr.	„	describe.	pt.	„	point.
diam.	„	diameter.	rect.	„	rectangle.
dist.	„	distance.	rectilin.	„	rectilineal.
ea. to ea.	„	each to each.	rem.	„	remainder.
equilat.	„	equilateral.	rt.	„	right.
ext.	„	exterior.	seg.	„	segment.
fig.	„	figure.	sq.	„	square.
hyp.	„	hypothesis.	str.	„	straight.
int.	„	interior.	viz.	„	namely.

SYMBOLS.

+	signifies	plus, the sign of addition.	≮	signifies	not less than.
−	„	minus, the sign of subtraction.	⊥	„	perpendicular to.
=	„	equal to.	∥	„	parallel to.
≠	„	not equal to.	∦	„	not parallel to.
>	„	greater than.	∵	„	because.
≯	„	not greater than.	∴	„	therefore.
<	„	less than.	⊙	„	circle.
			△	„	triangle.
			$\square^{m}$	„	parallelogram.
			∠	„	angle.

GEOMETRY.

DEFINITIONS.

I.

A point is that which has position, but not magnitude.

II.

A line is length without breadth.

III.

The extremities of a line are points.

IV.

A straight line is that which lies evenly between its extreme points.

V.

A superficies is that which has only length and breadth.

VI.

The extremities of a superficies are lines.

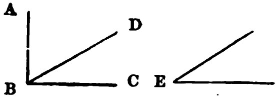
VII.

A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.

VIII.

A plane angle is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.

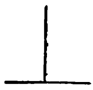
IX.



A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

N.B. When several angles are at one point B, any one of them is expressed by three letters, of which the letter that is at the vertex of the angle, that is, at the point in which the straight lines that contain the angle meet one another, is put between the other two letters, and one of these two is somewhere upon one of those straight lines, and the other upon the other line: Thus, the angle which is contained by the straight lines AB, CB, is named the angle ABC, or CBA; and that which is contained by DB, CB, is called the angle DBC or CBD; but if there be only one angle at a point, it may be expressed by a letter placed at that point, as the angle at E.

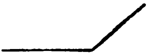
X.



When a straight line standing on another straight line makes the adjacent angles equal to one another, *each of the* angles is called a right angle; and the straight line which stands on the other, is called a perpendicular to it.

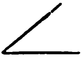
XI.

An obtuse angle is that which is greater than a right angle.



XII.

An acute angle is that which is less than a right angle.



XIII.

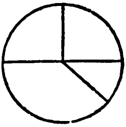
A term or boundary is the extremity of any thing.

XIV.

A figure is that which is enclosed by one or more boundaries.

XV.

A circle is a plane figure contained by one line, which is called the circumference, and is such, that all straight lines drawn from a certain point within the figure to the circumference, are equal to one another.



XVI.

And this point is called the centre of the circle.

XVII.

A diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

XVIII.

A semicircle is the figure contained by meter and the part of the circumferen off by the diameter.

XIX.

A segment of a circle is the figure contai a straight line, and the circumference off.

XX.

Rectilineal figures are those which ar tained by straight lines.

XXI.

Trilateral figures, or triangles, by three st lines.

XXII.

Quadrilateral, by four straight lines.

XXIII.

Multilateral figures, or polygons, by mor four straight lines.

XXIV.

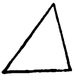
Of three-sided figures, an equilateral triangle is that which has three equal sides.

XXV.

An isosceles triangle is that which has only two sides equal.

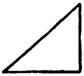
XXVI.

A scalene triangle is that which has three unequal sides.



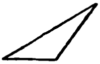
XXVII.

A right-angled triangle is that which has a right angle; and the side opposite the right angle is called the hypothenuse.



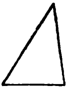
XXVIII.

An obtuse-angled triangle is that which has an obtuse angle.



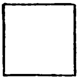
XXIX.

An acute-angled triangle is that which has three acute angles.



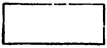
XXX.

Of four-sided figures, a square is that which has all its angles right angles, and all its sides equal.



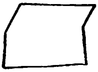
XXXI.

An oblong is that which has all its angles right angles, but has not all its sides equal.

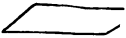


XXXII.

A rhombus is that which has all its sides equal, but its angles are not right angles.



XXXIII.



A rhomboid is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.

XXXIV.

All other four-sided figures, besides these, are called trapeziums.

XXXV.

Parallel straight lines are such as are in the same plane, and which being produced ever so far both ways, do not meet.

POSTULATES.

I.

Let it be granted, that a straight line may be drawn from any one point to any other point.

II.

That a terminated straight line may be produced to any length in a straight line.

III.

That a circle may be described from any centre, at any distance from that centre.

AXIOMS.

I.

Things which are equal to the same or equal things, are equal to one another.

II.

If equals be added to equals, the wholes are equal.

III.

If equals be taken from equals, the remainders are equal.

IV.

If equals be added to unequals, the wholes are unequal.

V.

If equals be taken from unequals, the remainders are unequal.

VI.

Things which are double of the same, or equal things, are equal to one another.

VII.

Things which are halves of the same, or equal things, are equal to one another.

VIII.

Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

IX.

The whole is greater than its part.

X.

Two straight lines cannot enclose a space.

XI.

All right angles are equal to one another.

XII.

If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together, less than two right angles, these straight lines, being continually produced, shall at length meet upon that side on which are the angles which are less than two right angles.

A proposition is something either proposed to be done, or to be demonstrated, and is either a problem or a theorem.

A problem is something proposed to be done; as the construction of a figure, &c.

A theorem is something proposed to be demonstrated.

A corollary is a consequence easily deduced from a proposition.

An axiom is an evident truth, but which admits of no demonstration.

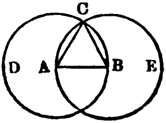
A postulate is a request to admit the possibility of performing an operation.

PROP. I. PROB. 1. 1 Eu.

To describe an equilateral triangle upon a given straight line.

Let AB be a given str. line, it is required to descr. an equilat. △ on AB.

From cr. A, at dist. AB, descr. ⊙ DBC; from cr. B, at dist. BA, descr. ⊙ ACE; from point C, in which the ⊙s cut each other, draw CA, CB. Then ACB is an equilat. △.



∵ A is cr. ⊙ BCD,
∴ AC = AB;
∵ B is cr. ⊙ ACE,
∴ BC = AB,
∴ AC = AB = BC.

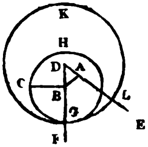
Wherefore on AB has been descr. an equilat. △ ACB.

PROP. II. PROB. 2. 1 Eu.

From a given point to draw a straight line equal to a given straight line.

Let A be the given pt., and BC the given str. line; it is required to draw from the pt. A a str. line = BC.

Post. 1. Draw **AB** ; on **AB** descr. an equilat. △
Prop. 1. **DAB** ; prod. **DA**, **DB** to **E** and **F** ; from cr. **B**
Post. 2. and dist. **BC**, descr. ⊙ **CGH** ; from cr. **D** and
Post. 3. dist. **DG**, descr. ⊙ **GKL**. Then **AL** = **BC**.



∵ B is cr. ⊙ CGH,
Def. 15. ∴ BC = BG ;
∵ D is cr. ⊙ GKL,
Def. 15. ∴ DL = DG ;
Constr. and part DA = part DB ;
Ax. 3. ∴ rem. AL = rem. BG ;
but BC = BG ;
Ax. 1. ∴ AL = BC.

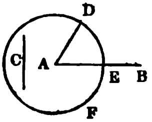
Wherefore, from the given pt. A, a str. line AL has been drawn equal to the given str. line BC.

PROP. III. PROB. 3. 1 Eu.

From the greater of two given straight line
to cut off a part equal to the less.

Let AB and C be the given str. lines, wher of AB > C ; it is required to cut off from A a part = C.

From A draw AD = C; from cr. A and dist. AD, descr. ⊙ DEF, cutting AB in E; then AE = C. Prop. 2. Post. 3.



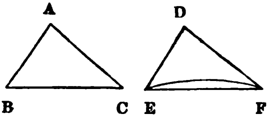
∵ A is cr. ⊙ DEF,
∴ AE = AD; Def. 15.
but C = AD; Constr.
∴ AE = C. Ax. 1.

Wherefore, from AB, the greater of two str. lines, a part AE has been cut off = C, the less.

PROP. IV. THEOR. 4. 1 Eu.

If two triangles have two sides of the one, equal to two sides of the other, each to each; and have likewise the angles contained by those sides equal to one another; they shall likewise have their bases, or third sides, equal, and the two triangles shall be equal; and their other angles shall be equal, each to each, viz., those to which the equal sides are opposite.

Let ABC, DEF, be two △s, of which AB = DE, AC = DF, and ∠ BAC = ∠ EDF. Then BC = EF, △ ABC = △ DEF, ∠ ABC = ∠ DEF, and ∠ ACB = ∠ DFE.



For if △ ABC be applied to △ D
that pt. A may be on D, and AB on D

Hyp. ∵ AB = DE,
∴ B coincides with E.
∴ { AB concides with DE,
Hyp. ∠ BAC = ∠ EDF ;
∴ AC falls upon DF.
Hyp. And ∵ AC = DF,
∴ C coincides with F.
But B coincides with E,
∴ BC coincides with EF.

For if BC do not coincide with EF, the
Ax. 10. str. lines enclose a space, which is impossi

∴ { BC coincides with and = EF,
△ ABC coincides with and = △ I
∠ ABC coincides with and = ∠ D
∠ ACB coincides with and = ∠ D

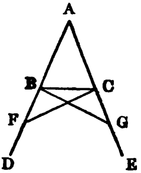
Therefore, if two triangles have, &c.

PROP. V. THEOR. 5

The angles at the base of an isosceles tr are equal to one another; and if the sides be produced, the angles upon the side of the base shall be equal.

Let ABC be an isosc. △, having AB :

Let AB and AC be prod. to D and E. Then $\angle$ ABC = $\angle$ ACB, and $\angle$ CBD = $\angle$ BCE.



In BD take any pt. F; from AE, the greater, cut off AG = AF; Prop. 3.

join FC, GB.

$$\because \left\{\begin{array}{ll} AF = AG, & \text{Constr.} \\ AB = AC, & \text{Hyp.} \end{array}\right.$$

and $\angle$ FAG common to $\triangle$s AFC, AGB,

$$\therefore \left\{\begin{array}{ll} \text{base } FC = \text{base } GB, & \\ \angle ACF = \angle ABG, & \\ \angle AFC = \angle AGB. & \text{Prop. 4.} \end{array}\right.$$

$$\begin{array}{rll} \text{Again, } \because \text{ whole } AF & = \text{whole } AG, & \text{Constr.} \\ \text{and part } AB & = \text{part } AC, & \text{Hyp.} \\ \therefore \text{ remain. } BF & = \text{remain. } CG; & \text{Ax. 3.} \end{array}$$

$$\text{and } \because \left\{\begin{array}{l} FC = GB, \\ BF = CG, \\ \angle BFC = \angle CGB; \end{array}\right.$$

$$\therefore \left\{\begin{array}{ll} \angle BCF = \angle CBG & \\ \angle FBC = \left\{\begin{array}{l} GCB, \text{ the } \angle\text{s on op-} \\ \text{posite side of base.} \end{array}\right. & \text{Prop. 4.} \end{array}\right.$$

$$\begin{array}{rll} \text{But whole } \angle ABG & = \text{whole } \angle ACF, & \\ \text{and part } \angle CBG & = \text{part } \angle BCF; & \text{Ax. 3.} \\ \therefore \text{ rem. } \angle ABC & = \left\{\begin{array}{l} \text{rem. } \angle ACB, \text{ the} \\ \angle\text{s at the base.} \end{array}\right. & \end{array}$$

∴ Therefore, the angles, &c.

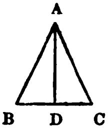
Cor.—Hence every equilateral △ is also equiangular.

PROP. V. THEOR.

OTHERWISE DEMONSTRATED.

The angles at the base of an isosceles triangle are equal to one another.

Let ABC be an isosc. △, having the side AB = side AC; then will ∠ ABC = ∠ ACB.



Let the str. line AD divide the ∠ BAC into two = parts.

∵ AD is common to both △s ABD, ACD,
and AB = AC,
Hyp. ∠BAD = ∠ DAC;

Prop. 4. ∴ △ ABD = △ ACD,
∠ ABC = ∠ ACB, which are the ∠s at the base.

Wherefore the angles at the base, &c.

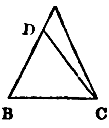
Note.—It is evident that some line, as AD, will bisect the ∠ BAC; and although the method of bisection is not known until Pro. 8 be solved, yet this does not affect the truth of the proposition ; since we are at liberty to suppose any line to be drawn, or any figure to be constructed, the existence of which does not involve an absurdity. The first proof of this proposition is that of Euclid, who, to avoid the *possibility* of the taking for granted that which may imply a contradiction, never supposes a thing to be done, the manner of doing which has not been previously explained.

The equality of the $\angle$s on the oppo. side of the base will follow from Prop. 12.

PROP. VI. THEOR. 6. 1 Eu.

If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to the equal angles, shall be equal to one another.

Let $\triangle$ ABC have $\angle$ ABC $=$ $\angle$ ACB ; then AB $=$ AC.



If AB $\neq$ AC,
one of them is $>$ the other ;
let AB $>$ AC ;
from AB cut off DB $=$ AC ; Prop. 3.
join DC.

In $\triangle$s DBC, ACB,

$$\because\left\{\begin{array}{l} DB = AC,\\ \angle DBC = \angle ACB,\\ BC \text{ common to both } \triangle s;\end{array}\right.$$

p. 4. $$\therefore\left\{\begin{array}{l} \text{base } DC = \text{base } AB,\\ \text{and } \triangle DBC = \triangle ACB,\end{array}\right.$$

$$\text{or, the less} = \left\{\begin{array}{l} \text{the greater, which}\\ \text{absurd;}\end{array}\right.$$

$\therefore$ AB is not $\neq$ AC,

i. e. AB = AC.

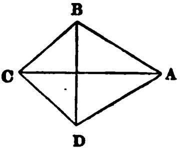
Wherefore, if two angles, &c.

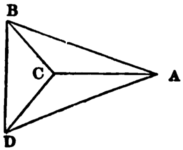
Cor.—Hence every equiangular triangle equilateral.

PROP. VII. THEOR.

If two triangles have three sides of the one respectively equal to the three sides of th other, each to each, the triangles are equa and the angles are equal which are opposit to the equal sides.

In $\triangle$s CBA, CDA, if AB, BC, CA = A DC, CA, respectively, ea. to ea. then $\triangle$ CB = $\triangle$ CDA, $\angle$CBA = $\angle$CDA, $\angle$BAC = CAD, $\angle$ ACB = $\angle$ ACD.





Conceive the $\triangle$s to be so applied to each other that the side CA may be common to both, and the equal sides BC of the one, and DC of the other, to terminate at the same pt. C, and to be on oppo. sides of CA. Join BD.

In $\triangle$ ABD $\because$ AB = AD, Hyp.

$\therefore \angle$ ABD = $\angle$ ADB; Prop. 5.

In $\triangle$ BCD $\because$ BC = CD, Hyp.

$\therefore \angle$ CBD = $\angle$ CDB, Prop. 5.

and $\angle$ CBA = $\angle$ CDA. Ax. 2, 3.

$$\because \left\{ \begin{array}{l} \text{AB, BC} = \text{AD, DC, ea. to ea.} \\ \text{and incl. } \angle \text{CBA} = \text{incl. } \angle \text{CDA}; \end{array} \right.$$ Hyp.

$$\therefore \left\{ \begin{array}{l} \triangle \text{CBA} = \triangle \text{CDA}, \\ \angle \text{BAC} = \angle \text{CAD}, \\ \angle \text{ACB} = \angle \text{ACD}. \end{array} \right.$$ Prop. 4.

Wherefore, if two triangles, &c.

PROP. VIII. PROB. 9. 1 Eu.

To bisect a given rectilineal angle, that is, to divide it into two equal angles.

Let BAC be the given rectilin. $\angle$; it is required to bisect it.

Take any point D in AB.

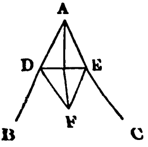
From AC cut off AE = AD, Prop. 3.

join DE.

On DE descr. equilat. $\triangle$ DEF, Prop. 1.

join AF.

Then AF bisects $\angle$ BAC.



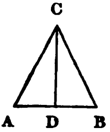
Constr. AD = AE,
∵ AF common to △s DAF, EAF,
Constr. DF = EF ;
Prop. 7. ∴ ∠ DAF = ∠ EAF.

Wherefore the rectilin. ∠ BAC is bisected by AF.

PROP. IX. PROB. 10. 1 Eu.

To bisect a given finite straight line, that is, to divide it into two equal parts.

To divide AB into two = parts.
Prop. 1. Upon AB descr. equilat. △ ABC.
Prop. 8. Bisect ∠ ACB by CD.
Then AB is bisected at D.



Constr. AC = CB,
∵ CD common to △s ACD, BCD,
Constr. ∠ ACD = ∠ BCD ;
Prop. 4. ∴ base AD = base DB.

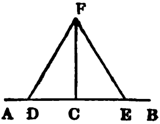
Therefore the str. line AB is divided into two = parts in the point D.

PROP. X. PROB. 11. 1 Eu.

To draw a straight line at right angles to a given straight line, from a given point in the same.

Let AB be the given line, and C a given pt. in it; it is required to draw from C a str. line ⊥ AB.

Take any point D in AC, make CE = CD. Prop. 3.
On DE descr. equilat. △ DFE; join FC. Prop. 1.
Then FC ⊥ AB.



∵ { FC is common to △s DCF, ECF,
FD, DC = FE, CE, ea. to ea. Constr.
∴ { ∠ DCF = ∠ ECF, which are adj. ∠ s, Prop. 7.
and FC ⊥ AB. Def. 10.
Therefore there has been drawn, &c.

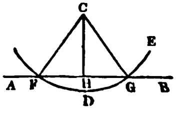
PROP. XI. PROB. 12. 1 E.

To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.

Let AB be the given line, and C a pt. without it; it is required to draw CH ⊥ AB.

Take any pt. D upon the other side of AB.
From cent. C, dist. CD, descr. ⊙ EGDF Post. 3.
meeting AB in F and G; bisect FG in H; join Prop. 9.
CH.

Then CH ⊥ AB.



Join CF, CG;

Const. Def. 15. ∵ {FH, CF = HG, CG, ea. to ea.
HC common to △s FHC, GHC;

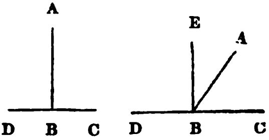
Prop. 7. Def. 10. ∴ {∠ CHF = ∠ CHG, and are adj. ∠s,
and CH ⊥ AB.

Therefore, the ⊥ CH has been drawn, &c.

PROP. XII. THEOR. 13. 1 Eu.

The angles which one straight line makes with another upon one side of it, are either two right angles, or are together equal to two right angles.

Let AB make with DC, on the same side of it, ∠s DBA, ABC; then ∠ DBA + ∠ ABC = 2 rt. ∠s.



If ∠ ABC = ∠ DBA,

Def. 10. each of them is a rt. ∠ :

Prop. 10. But if ∠ ABC ≠ ∠ DBA, draw BE ⊥ DC;

Def. 10. ∴ ∠s CBE, DBE are rt. ∠s.

and ∵ ∠ CBE = ∠ ABC + ∠ ABE;

add to these equals ∠ DBE;

Ax. 2. ∴ ∠CBE + ∠DBE = $\left\{\begin{array}{l}\angle \text{ ABC} + \angle \text{ ABE} + \\ \angle \text{ DBE}:\end{array}\right.$

Again ∵ ∠ DBA = ∠ DBE + ∠ ABE,

add to these equals ∠ ABC;

$\therefore \angle DBA + \angle ABC = \left\{ \begin{array}{l} \angle ABC + \angle ABE + \\ \quad \angle DBE : \end{array} \right.$ Ax. 2.

but it has been shown that

$\angle CBE + \angle DBE =$ same 3 $\angle$s;

$\therefore \angle CBE + \angle DBE = \angle DBA + \angle ABC$; Ax. 1.

but $\angle CBE + \angle DBE = 2$ rt. $\angle$s,

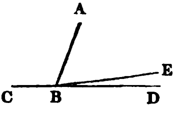
$\therefore \angle DBA + \angle ABC = 2$ rt. $\angle$s. Ax. 1.

Therefore the angles, &c.

PROP. XIII. THEOR. 14. 1 Eu.

If at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

At the point B in AB let BC, BD, on the oppo. sides of AB, make the adj. $\angle$s ABC + $\angle$ABD = 2 rt. $\angle$s; then shall BD be in the same straight line with BC.



If BD be not in the same str. line with CB, let BE be in the same str. line with it; then $\because$ str. line AB makes with str. line CBE, on the same side of it, the $\angle$s ABC, ABE:

$\therefore \angle ABC + \angle ABE = 2$ rt. $\angle$s; Prop

Hyp. but $\angle ABC + \angle ABD = 2$ rt. $\angle$;

Ax. 1. $\therefore \angle ABC + \angle ABE = \angle ABC + \angle ABD$.

From these equals take away $\angle$ ABC,

Ax. 3. $\therefore \angle ABE = \angle ABD$;

$\text{or, less} = \left\{\begin{array}{l}\text{greater, which is}\\ \quad\text{absurd;}\end{array}\right.$

$\therefore$ BE is not in the same str. line with BC.

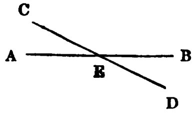
In like manner it may be shown, that no other line but BD can be in the same str. line with BC.

Wherefore, if at a point, &c.

PROP. XIV. THEOR. 15. 1 Eu.

If two straight lines cut one another, the vertical or opposite angles shall be equal.

Let the str. lines AB, CD, cut one another in E; then $\angle$ AEC $=$ $\angle$ DEB, and $\angle$ CEB $=$ $\angle$ AED.



$\because$ str. line AE makes with CD the $\angle$s CEA, AED,

Prop. 12. $\therefore \angle CEA + \angle AED = 2$ rt. $\angle$s.

Again,

$\because$ str. line DE makes with AB the $\angle$s AED, DEB.

Prop. 12. $\therefore \angle AED + \angle DEB = 2$ rt. $\angle$s,

Ax. 1. $\therefore \angle CEA + \angle AED = \angle AED + \angle DEB$.

From these equals take away the common
∠ AED,
∴ remain. ∠ CEA = remain. ∠ DEB. Ax. 3.
In the same manner it may be shown, that
∠ CEB = ∠ AED.

Therefore, if two straight lines, &c.

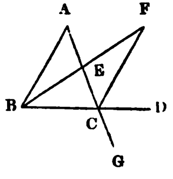
Cor. 1.—If two str. lines cut one another, the ∠s which they make at the pt. where they cut, are together equal to 4 rt. ∠s.

Cor. 2.—All the angles made by any number of lines meeting in one pt., are together equal to 4 rt. ∠s.

PROP. XV. THEOR. 16. 1 Eu.

If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

Let ABC be a △, and the side BC be prod. to D; then the ext. ∠ ACD > ∠ CBA or ∠ BAC, the int. oppo. ∠s.



Bisect AC in E. Prop. 9

Join BE and prod. it to F.
C

Prop. 3. Make $EF = BE$.

Join FC.

Constr. $\because \begin{cases} AE = EC, \\ BE = EF, \\ \angle AEB = \angle CEF; \end{cases}$ Prop. 14.

Prop. 4. $\therefore \begin{cases} \text{base } AB = \text{base } CF, \\ \angle BAC = \angle ECF, \end{cases}$

Ax. 9. but $\angle ACD > \angle ECF$;

$\therefore \angle ACD > \angle BAC$.

In the same way, if BC be bisected, and AC be prod. to G, it may be shown, that

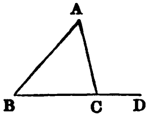
Prop. 14. $\angle BCG$ or $\angle ACD > \angle ABC$.

Therefore, if one side, &c.

PROP. XVI. THEOR. 17. 1 Eu.

Any two angles of a triangle are together less than two right angles.

Let ABC be any △, any two of its ∠s are together less than 2 rt. ∠s.



Produce BC to D.

Prop. 15. $\because$ ext. $\angle ACD > \angle ABC$, the int. and oppo. $\angle$;

add $\angle ACB$ to each,

$\therefore \angle ACD + \angle ACB > \angle ABC + \angle ACB$;

Prop. 13. but $\angle ACD + \angle ACB = 2$ rt. ∠s,

$\therefore \angle ABC + \angle ACB < 2$ rt. ∠s.

In like manner it may be proved, that

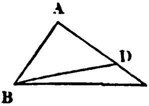
$\angle$ BAC + $\angle$ ACB $<$ 2 rt. $\angle$s,
and $\angle$ CAB + $\angle$ ABC $<$ 2 rt. $\angle$s.

Therefore, any two angles, &c.

PROP. XVII. THEOR. 18. 1 Eu.

The greater side of every triangle is opposite to, or subtends the greater angle.

If ABC be a $\triangle$ of which the side AC $>$ side AB, then $\angle$ ABC $>$ $\angle$ ACB.



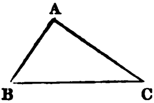
$\because$ AC $>$ AB,
Make AD = AB, Prop. 3
join BD;
then, $\because$ AB = AD, Constr.
$\therefore$ $\angle$ ABD = $\angle$ ADB; Prop. 5.
but ext. $\angle$ ADB $>$ int. $\angle$ ACB, Prop. 15.
$\therefore$ $\angle$ ABD $>$ $\angle$ ACB,
much more is $\angle$ ABC $>$ $\angle$ ACB.

Therefore the greater side, &c.

PROP. XVIII. THEOR. 19. 1 Eu.

The greater angle of every triangle is subtended by the greater side; or has the greater side opposite to it.

If ABC be a △, of which ∠ ABC > ∠
then $AC > AB$.



For AC must be either >, =, or <

1st. If $AC = AB$

Prop. 5. then ∠ ABC = ∠ ACB.

By Hyp. which is not true;

$\therefore \quad AC \neq AB$.

2dly. If $AC < AB$,

Prop. 17. then ∠ ABC < ∠ ACB,

By Hyp. which is not true.

$\therefore \quad AC \nless AB$.

It has been shown that

$AC \neq AB$,

$\therefore \quad AC > AB$.

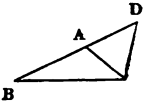
Therefore the greater ∠, &c.

PROP. XIX. THEOR. 20.

Any two sides of a triangle are to
greater than the third side.

Let ABC be a △, then

$$AB + AC > BC,$$
$$AB + BC > AC,$$
$$BC + AC > AB.$$



Produce BA to D.

Make $AD = AC$. Prop. 3.

Join DC.

Then $\because$ $DA = AC$,

$\therefore$ $\angle ADC = \angle ACD$; Prop. 5.

but $\angle BCD > \angle ACD$. Ax. 9.

$\therefore$ $\angle BCD > \angle ADC$.

In $\triangle DCB$ $\because$ $\angle BCD > \angle BDC$,

$\therefore$ $BD > BC$; Prop. 18.

but $BD = AB + AC$, Constr.

$\therefore$ $AB + AC > BC$.

Similarly, $\begin{cases} AB + BC > AC, \\ BC + AC > AB. \end{cases}$

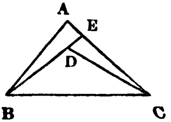
Wherefore any two sides, &c.

PROP. XX. THEOR. 21. 1Eu.

If from the ends of the side of a triangle, there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

Let ABC be a $\triangle$; from B, C, the ends of the side BC, draw BD, CD, from the point D within the $\triangle$; then

$BD + DC < BA + AC$, and $\angle BDC > \angle BAC$.



Prod. BD to E.
Prop. 19. In $\triangle$ ABE, $\because$ $BA + AE > BE$,
add EC to each;
$\therefore$ $BA + AC > BE + EC$.
Prop. 19. In $\triangle$ CED, $\because$ $CE + ED > CD$,
add DB to each;
$\therefore$ $EC + BE > CD + DB$.
But it has been proved $BA + AC > BE + EC$,
much more then is $BA + AC > CD + DB$.
Prop. 15. Again, in $\triangle$ CDE, ext. $\angle$ BDC > int. $\angle$ CED,
and in $\triangle$ ABE, ext. $\angle$ CED > int, $\angle$ BAC,
much more then is $\angle$ BDC > $\angle$ BAC.

Therefore, if from the ends of, &c.

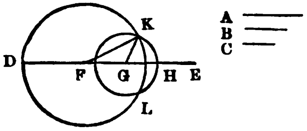
PROP. XXI. PROB. 22. 1 Eu.

To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these must be greater than the third. (19 *Prop.*)

Let A, B, C, be the 3 given straight lines. It is required to make a $\triangle$, of which the sides shall be equal to A, B, C, respectively.

From D take DE unlimited towards E.
Prop. 3. Make $DF = A$,
$FG = B$,
$GH = C$.
Post. 3. From cent. F, and dist. FD, descr. $\odot$ DKL.
From cent. G, and dist. GH, descr. $\odot$ HLK.
Join KF and KG.

Then sides of △ KFG = A, B, C.



∵ F is cent. ⊙ DKL;
∴ FD = FK; Def. 15.
but FD = A, Constr.
∴ FK = A. Ax. 1.
Also ∵ G is cent. ⊙ LKH,
∴ GH = GK; Def. 15.
but GH = C, Constr.
∴ GK = C. Ax. 1.
and FG = B.

∴ sides of △ KFG are made = A, B, C, which was to be done.

PROP. XXII. PROB. 23. 1 Eu.

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given str. line, A a given pt. in it, DCE the given ∠; to make an ∠ at A in AB = ∠ DCE.

In CD, CE, take any points D, E.

Join DE.

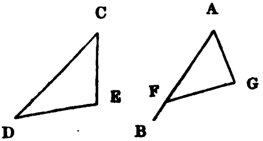
Make △ AFG, so that

CD = AF,

CE = AG,

Prop. 21. DE = FG;

then ∠ FAG = ∠ DCE.



Constr. ∵ CD, CE, DE = AF, AG, FG, ea. to ea.

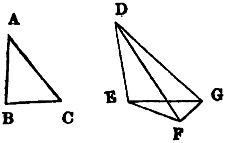
Prop. 7. ∴ ∠ DCE = ∠ FAG.

Therefore, at the given point A in AB, the ∠ angle FAG is made = given ∠ DCE, which was to be done.

PROP. XXIII. THEOR. 24. 1 Eu.

If two triangles have two sides of the one, equal to two sides of the other, each to each, but the angle contained by the two sides of one of them, greater than the angle contained by the two sides equal to them, of the other; the base of that which has the greater angle shall be greater than the base of the other.

Let ABC, DEF, be two △s, having the sides AB, AC = sides DE, DF, ea. to ea., but ∠ BAC > ∠ EDF: then, base BC > base EF.



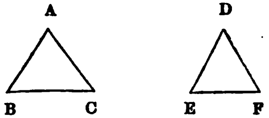
Of the sides DE, DF, let DE ≯ DF,
at D in the str. line DE,
make { ∠ EDG = ∠ BAC, P
DG = AC or DF ; P
join EG, GF.
∵ { AB, AC = ED, DG, ea. to ea. H
∠ BAC = ∠ EDG, C
∴ BC = EG, P
and ∵ DG = DF, C
∴ ∠ DFG = ∠ DGF ; P
but ∠ DGF > ∠ EGF, A
∴ ∠ DFG > ∠ EGF,
∴ much more is ∠ EFG > ∠ EGF,
∴ EG > EF ; P
but it was proved that EG = BC,
∴ BC > EF.

Therefore, if two triangles, &c.

PROP. XXIV. THEOR. 25. 1 Eu.

If two triangles have two sides of the one, equal to two sides of the other, each to each, but the base of the one, greater than the base of the other ; the angle contained by the sides of that which has the greater base, shall be *greater than the angle contained by the sides equal to them, of the other.*

Let ABC, DEF, be two Δs, having the sides AB, AC $=$ sides DE, DF, ea. to ea., and base BC $>$ base EF; then $\angle$ BAC $>$ $\angle$ EDF.



For
$\angle$ BAC must be either $>$, $=$, or $<$ $\angle$ EDF.
1st. If $\angle$ BAC $=$ $\angle$ EDF,
4. $\therefore$ BC $=$ EF,
which is not true;
$\therefore$ $\angle$ BAC $\neq$ $\angle$ EDF.
2dly. If $\angle$ BAC $<$ $\angle$ EDF,
23. $\therefore$ base BC $<$ base EF,
which is not true;
$\therefore$ $\angle$ BAC $\nless$ $\angle$ EDF;
and it has been proved $\angle$ BAC $\neq$ $\angle$ EDF,
$\therefore$ $\angle$ BAC $>$ $\angle$ EDF.

Wherefore, if two triangles, &c.

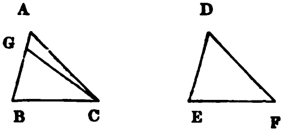
PROP. XXV. THEOR. 26. 1 Eu.

If two triangles have two angles of the one, equal to two angles of the other, each to each, and one side equal to one side; viz. *either the sides adjacent to the equal angles, or the sides opposite to equal angles in each; then* *shall the other sides be equal, each to each, and also the third angle of the one, equal to the third angle of the other.*

CASE I. When sides adj. to equal $\angle$s in each $\triangle$ be equal.

In $\triangle$s ABC, DEF,

$$\text{let}\left\{\begin{array}{l} BC = EF, \\ \angle ABC = \angle DEF, \\ \angle BCA = \angle EFD, \end{array}\right\}\text{then shall}\left\{\begin{array}{l} AB = DE, \\ AC = DF, \\ \angle BAC = \angle EDF. \end{array}\right.$$



For if $AB \neq DE$, then one is $>$ other;
let $AB > DE$, make $BG = DE$, join GC. Prop. 3.

$$\because\left\{\begin{array}{l} GB, BC = DE, EF, \text{ ea. to ea.} \\ \angle GBC = \angle DEF; \end{array}\right.$$

$$\therefore\left\{\begin{array}{l} \text{base } GC = \text{base } DF, \\ \triangle GBC = \triangle DEF, \\ \angle GCB = \angle DFE; \end{array}\right. \qquad \text{Prop. 4.}$$

but $\angle BCA = \angle DFE$, Hyp.

$\therefore$ $\angle GCB = \angle BCA$, Ax 1.

i. e. less $=$ greater,
which is absurd;

$\therefore$ AB is not $\neq$ DE,

i e. $AB = DE$.

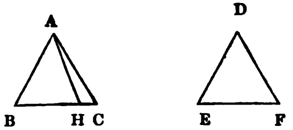
Then

$$\because\left\{\begin{array}{l} AB, BC = DE, EF, \text{ ea. to each,} \\ \angle ABC = \angle DEF; \end{array}\right. \qquad \text{Hyp.}$$

Prop. 4. $\therefore \begin{cases} \text{base AC} = \text{base DF}, \\ \angle\,\text{BAC} = \angle\,\text{EDF}. \end{cases}$

CASE 2. When sides oppo. equal $\angle$s in each $\triangle$ are equal to each other.

$$\text{Let}\begin{cases} \text{AB} = \text{DE}, \\ \angle\,\text{ABC} = \angle\,\text{DEF}, \\ \angle\,\text{BCA} = \angle\,\text{EFD}, \end{cases} \text{then shall} \begin{cases} \text{AC} = \text{DF}, \\ \text{BC} = \text{EF}, \\ \angle\,\text{BAC} = \angle\,\text{EDF}, \end{cases}$$



For if BC $\neq$ EF,

let BC $>$ EF.

Prop. 3. Make BH $=$ EF;

join AH.

$$\because \begin{cases} \text{AB, BH} = \text{DE, EF, ea. to ea.} \\ \angle\,\text{ABC} = \angle\,\text{DEF}, \end{cases}$$

Prop. 4. $\therefore \begin{cases} \text{base AH} = \text{base DF}, \\ \triangle\,\text{ABH} = \triangle\,\text{DEF}, \\ \angle\,\text{BHA} = \angle\,\text{EFD}; \end{cases}$

Hyp. but $\angle$ BCA $=$ $\angle$ EFD,

Ax. 1. $\therefore$ $\angle$ BHA $=$ $\angle$ BCA,

i. e. ext. $\angle$ of $\triangle$ AHC, $\angle$ BHA $=$ $\angle$ BCA, the int. and oppo. $\angle$,

Prop. 15. which is impossible,

$\therefore$ BC is not $\neq$ EF,

i. e. BC $=$ EF.

Hyp. $\text{Then} \because \begin{cases} \text{AB, BC} = \text{DE, EF, ea. to ea.} \\ \angle\,\text{ABC} = \angle\,\text{DEF}; \end{cases}$

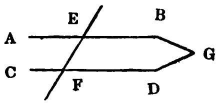
Prop. 4. $\therefore \begin{cases} \text{base AC} = \text{base DF}, \\ \angle\,\text{BAC} = \angle\,\text{EDF}. \end{cases}$

Therefore, if two triangles, &c.

PROP. XXVI. THEOR. 27. 1 Eu.

If a straight line falling upon two other straight lines, make the alternate angles equal to one another, these two straight lines shall be parallel.

Let str. line EF, which falls upon the str. lines AB, CD, make alt. $\angle AEF =$ alt. $\angle EFD$: then shall $AB \parallel CD$.



For if $AB \nparallel CD$, they will, when produced, meet either towards B, D, or towards A, C. Def. 35.

Suppose the former, and let them meet in G, then GEF will form a $\triangle$;

$\therefore$ ext. $\angle AEF >$ int. $\angle EFG$; Prop. 15.

but $\angle AEF =$ $\angle EFG$ or $\angle EFD$, Hyp.

which is impossible.

$\therefore$ AB, CD, being produced, do not meet towards B, D.

In like manner it may be proved that they do not meet towards A, C;

$\therefore AB \parallel CD$; Def. 35.

Wherefore if a str. line, &c.

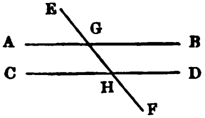
PROP. XXVII. THEOR. 28. 1 Eu.

If a straight line falling upon two other straight lines, make the exterior angle equal to the interior and opposite upon the same side of the line; or make the interior angles upon the same side together equal to two right angles; the two straight lines shall be parallel to one another.

Let the str. line EF, which falls upon AB and CD, make ext. $\angle$ EGB = $\angle$ GHD, the int. and oppo. $\angle$ on the same side.

Or int. $\angle$s on the same side, viz.

$\angle$ BGH + $\angle$ GHD = 2 rt. $\angle$s;
then AB $\parallel$ CD.



Hyp For $\because$ $\angle$EGB = $\angle$GHD,
Prop. 14. and $\angle$EGB = $\angle$AGH;
Ax 1. $\therefore$ $\angle$GHD = $\left\{\begin{array}{l}\angle \text{AGH, which are} \\ \text{alt. } \angle\text{s,}\end{array}\right.$
Prop. 26. $\therefore$ AB $\parallel$ CD.

Again,
Hyp. $\because$ $\angle$BGH + $\angle$GHD = 2 rt. $\angle$s,
Prop. 12. and $\angle$AGH + $\angle$BGH = 2 rt. $\angle$s,
Ax. 1. $\therefore$ $\angle$BGH + $\angle$GHD = $\angle$AGH + $\angle$BGH;

from these equals take $\angle$ BGH,

$\therefore \quad \angle GHD = \left\{ \begin{array}{l} \angle AGH, \text{ which are} \\ \text{alt. } \angle\text{s}, \end{array} \right.$ Ax. 3.

$\therefore \quad AB \parallel CD.$ Prop. 26.

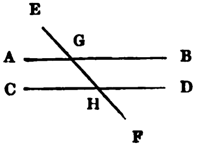
Wherefore, if a straight line, &c.

PROP. XXVIII. THEOR. 29. 1 Eu.

If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another; and the exterior angle equal to the interior and opposite angle on the same side; and likewise the two interior angles upon the same side together equal to two right angles.

Let the str. line EF fall on the $\parallel$ str. lines AB, CD, then

alt. $\angle$ AGH = alt. $\angle$ GHD,
ext. $\angle$ EGB = int. $\angle$ GHD,
And $\angle$ BGH + $\angle$ GHD = 2 rt. $\angle$s.



For if $\quad \angle AGH \neq \angle GHD,$
let $\quad \angle AGH > \angle GHD,$
add to each the $\angle$ BGH,
$\therefore \ \angle AGH + \angle BGH > \angle BGH + \angle GHD;$

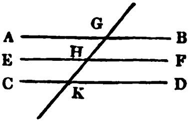
Prop. 12. but $\angle AGH + \angle BGH = 2$ rt. $\angle$s,
$\therefore$ $\angle BGH + \angle GHD < 2$ rt. $\angle$s.
Ax. 12. $\therefore$ AB, CD, will meet if prod. far enough;
Hyp. but they never meet, since they are parallel;
$\therefore$ $\angle AGH$ is not $\neq \angle GHD$,
i. e. $\angle AGH = \angle GHD$,
Prop. 14. but $\angle AGH = \angle EGB$,
Ax. 1. $\therefore$ $\angle EGB = \angle GHD$;
add to each the $\angle$ BGH,
Ax. 2. $\therefore$ $\angle EGB + \angle BGH = \angle BGH + \angle GHD$,
Prop. 12. but $\angle EGB + \angle BGH = 2$ rt. $\angle$s,
$\therefore$ $\angle BGH + \angle GHD = 2$ rt. $\angle$s.

Wherefore, if a straight line, &c.

PROP. XXIX. THEOR. 30. 1 Eu.

Straight lines which are parallel to the same straight line, are parallel to each other.

Let AB, CD, be each $\parallel$ EF; then shall AB $\parallel$ CD.



Let str. line GK cut AB, EF, CD, in the points G, H, K.

Then $\because$ GK cuts the $\parallel$ lines AB, EF,
Prop. 28. $\therefore$ $\angle AGH = \angle GHF$.

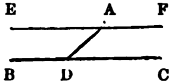
Again, ∵ GK cuts the ∥ lines EF, CD,
∴ ∠ GHF = ∠ GKD, Pr
and it was proved ∠ AGH = ∠ GHF;
∴ ∠ AGH = ∠ GKD, Ax.
which are alt. ∠ s.
∴ AB ∥ CD. Prop
Wherefore str. lines, &c.

PROP. XXX. PROB. 31. 1 Eu.

To draw a straight line through a given point, parallel to a given straight line.

Let A be the given point, BC the given str. line. To draw a str. line through A ∥ BC.

In BC take any pt. D,
join AD,
make ∠ DAE = ∠ ADC, Prop. 22.
prod. EA to F,
then shall EF ∥ BC.



∵ AD meets BC, EF, and makes
alt. ∠ EAD = alt. ∠ ADC,
∴ EF ∥ BC. Prop. 26.

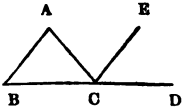
Therefore, a str. line has been drawn through A ∥ BC.

PROP. XXXI. THEOR. 32. 1 Eu.

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are together equal to two right angles.

Let ABC be a $\triangle$, and the side BC be prod. to D. Then shall the ext. $\angle$ ACD = $\angle$ CAB + $\angle$ ABC, the two int. and oppo. $\angle$s. And the 3 int. $\angle$s, viz.,

$$\angle ABC + \angle BCA + \angle CAB = 2 \text{ rt. } \angle\text{s.}$$



Draw CE $\parallel$ AB;

30. $\because$ AC meets them,

28. $\therefore$ alt. $\angle$BAC = alt. $\angle$ACE.

Again, $\because$ BD falls upon the $\parallel$ lines AB, CE,

28. $\therefore$ ext. $\angle$ ECD = $\left\{\begin{array}{l}\angle \text{ ABC, the int.} \\ \text{and oppo. } \angle\text{;}\end{array}\right.$

but $\angle$ ACE = $\angle$BAC.

$\therefore$ whole ext. $\angle$ACD = $\left\{\begin{array}{l}\angle \text{ BAC } + \angle \\ \text{ABC,}\end{array}\right.$

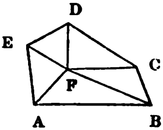
to each of these equals add $\angle$ ACB,

$\therefore$ $\angle$ ACD + $\angle$ACB = $\left\{\begin{array}{l}\angle\text{ACB} + \angle \\ \text{BAC} + \angle \\ \text{ABC;}\end{array}\right.$

but $\angle ACD + \angle ACB = 2$ rt. $\angle$s. Prop. 12.
$\therefore \angle ACB + \angle BAC + \angle ABC = 2$ rt. $\angle$s. Ax. 1.

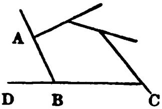
Wherefore, if a side of a triangle, &c.

Cor. 1.—All the int. $\angle$s of any rectilin. figure, together with 4 rt. $\angle$s, are equal to twice as many rt. $\angle$s as the figure has sides.



For any rectilin. figure ABCDE can be divided into as many $\triangle$s as the figure has sides, by drawing str. lines from a pt. F within the figure to each of its angles. Then, by the preceding proposition, all the $\angle$s of these $\triangle$s are equal to twice as many rt. $\angle$s as there are $\triangle$s, *i. e.* as there are sides of the figure: and the same $\angle$s are equal to the $\angle$s of the figure, together with the $\angle$s at the pt. F, which is the common vertex of the $\triangle$s; that is, together with 4 rt. $\angle$s. Cor. 2, Prop. 14.

Cor. 2.—All the ext. $\angle$s of any rectilin. figure are together equal to 4 rt. $\angle$s.



$$\because \left.\begin{array}{l}\text{every int. } \angle ABC + \\ \text{adj. ext. } \angle ABD\end{array}\right\} = 2 \text{ rt. } \angle\text{s},$$ Prop. 13.

$$\therefore \left.\begin{array}{l}\text{all the int. }\angle\text{s}\\ +\text{all the ext. }\angle\text{s}\end{array}\right\} = \left\{\begin{array}{l}\text{twice as many rt. }\angle\text{s as}\\ \text{there are sides.}\end{array}\right.$$

$$\text{but} \left.\begin{array}{l}\text{all the int. }\angle\text{s}\\ +\ 4\text{ rt. }\angle\text{s}\end{array}\right\} = \left\{\begin{array}{l}\text{twice as many rt. }\angle\text{s as}\\ \text{there are sides (by last}\\ \text{Cor.),}\end{array}\right.$$

Ax. 1. $$\therefore \left.\begin{array}{l}\text{all the int. }\angle\text{s}\\ +\text{all the ext. }\angle\text{s}\end{array}\right\} = \left\{\begin{array}{l}\text{all the int. }\angle\text{s} + 4\text{ rt.}\\ \angle\text{s;}\end{array}\right.$$

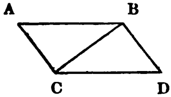
from these equals take away all the int. ∠s,

∴ all the ext. ∠s = 4 rt. ∠s.

PROP. XXXII. THEOR. 33. 1 Eu.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are also themselves equal and parallel.

Let AB, CD, be = and ∥ str. lines, joined towards the same parts by the str. lines AC, BD : then shall AC and BD be = and ∥ to each other.



Join BC.

∵ AB ∥ CD and BC meets them,

Prop. 28. ∴ alt. ∠ ABC = alt. ∠ BCD;

Hyp. $$\because \left\{\begin{array}{l}\text{AB} = \text{CD,}\\ \text{BC common to }\triangle\text{s ABC, DCB,}\\ \angle\text{ ABC} = \angle\text{ BCD,}\end{array}\right.$$

$\therefore \begin{cases} \text{base AC} = \text{base BD}, \\ \angle \text{ACB} = \angle \text{CBD}, \end{cases}$ Prop. 4.

$\because \begin{cases} \text{BC meets AC, BD, and makes} \\ \text{alt.} \angle \text{ACB} = \text{alt.} \angle \text{CBD}, \end{cases}$

$\therefore$ AC $\parallel$ BD: Prop. 26.

and it has been proved that

AC = BD.

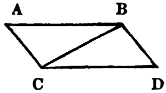
Therefore str. lines, &c.

PROP. XXXIII. THEOR. 34. 1 Eu.

The opposite sides and angles of parallelograms are equal to one another, and the diameter bisects them, that is, divides them into two equal parts.

N.B.—A parallelogram is a four-sided figure, of which the opposite sides are parallel; and the diameter is the straight line joining two of its opposite angles.

Let ACDB be a ▱^{m}, of which BC is a diam. the opp. sides and ∠s of the figure are = to one another; and BC bisects the figure.



For

$\because$ AB $\parallel$ CD,

and BC meets them,

$\therefore$ alt. ∠ ABC = alt. ∠ BCD. Prop. 28.

$\because$ AC $\parallel$ BD, and BC meets them,

$\therefore$ alt. ∠ ACB = alt. ∠ CBD. Prop 28

$\therefore$ in $\triangle$s ABC, CBD,
$\because$ $\angle$s ABC, BCA = $\angle$s BCD, CBD, ea. to ea.
and adjacent side BC common to both $\triangle$s,

Prop. 25. $\therefore \begin{cases} AB, AC = CD, BD, \text{ ea. to ea.} \\ \angle BAC = \angle BDC, \end{cases}$

and $\because \begin{cases} \angle ABC = \angle BCD. \\ \angle CBD = \angle ACB, \end{cases}$

Ax. 2. $\therefore$whole $\angle$ ABD = whole $\angle$ ACD;
and it has been proved

$\angle$ BAC = $\angle$ BDC.

$\therefore$ the opp. sides and $\angle$s of ▱$^{\text{ms}}$are = to one another.

Again, in the $\triangle$s ABC, CBD,

$\because \begin{cases} AB = CD, \\ BC \text{ common to both } \triangle\text{s}, \\ \angle ABC = \angle BCD, \end{cases}$

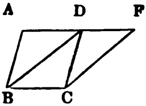
Prop. 4. $\therefore$ $\triangle$ ABC = $\triangle$ BCD.
$\therefore$ diam. BC divides the ▱$^{\text{m}}$ ACDB into two = parts.

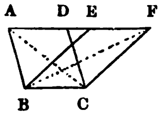
Wherefore the opposite sides and angles, &c.

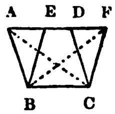
PROP. XXXIV. THEOR. 35. 1 Eu.

Parallelograms upon the same base, and between the same parallels, are equal to one another.

Let the ▱$^{\text{ms}}$ ABCD, EBCF, be upon the same base BC, and between the same ||s, AF, BC; then ▱$^{\text{m}}$ ABCD = ▱$^{\text{m}}$EBCF.







1. If the sides AD, DF, of the ▱^{ms} ABCD, DBCF, opp. to BC the base, be terminated in the same point D [as in fig. 1],

∵ each ▱^{m} = double the △ BDC, Prop. 33.

∴ the ▱^{m} ABCD = ▱^{m} DBCF. Ax. 6.

2. If the sides AD, EF, be not terminated in the same point (figs. 2 and 3),

then ∵ ABCD, EBCF are ▱^{ms},

$$\therefore \left\{\begin{array}{l} AD = BC, \\ EF = BC; \end{array}\right.$$ Prop. 33.

∴ AD = EF, Ax. 1.

and DE is common to both,

$$\therefore \left\{\begin{array}{l} \text{the whole or} \\ \text{remain. AE} \end{array}\right\} = \text{whole or remain. DF.}$$ Ax. 2, 3.

$$\text{And} \because \left\{\begin{array}{l} \text{EA, AB} = \text{FD, DC, ea. to ea.} \\ \text{ext.} \angle \text{ FDC} = \text{int.} \angle \text{ EAB,} \end{array}\right.$$ Prop. 33. Prop. 28.

$$\therefore \left\{\begin{array}{l} \text{base EB} = \text{base FC,} \\ \triangle \text{ EAB} = \triangle \text{ FDC.} \end{array}\right.$$ Prop. 4.

Take the △ EAB from the trapezium ABCF; and from the same trapezium take the △ FDC; then the remainders are =; that is, Ax. 3.

▱^{m} ABCD = ▱^{m} EBCF.

Wherefore parallelograms, &c.

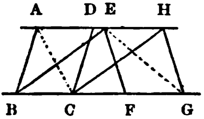
Cor.—Triangles upon the same base, and between the same parallels, are equal to one another. For if the diameters AC, FB, be drawn (figs. 2 and 3), the △s ABC, FBC,

are the halves of the equal ▱$^{\text{ms}}$ ABCD, EBCF,
Ax. 7. ∴ the △s are equal to one another.

PROP. XXXV. THEOR. 36. 1 Eu.

Parallelograms upon equal bases, and between the same parallels, are equal to one another.

Let ABCD, EFGH, be ▱$^{\text{ms}}$ upon = bases BC, FG, and between the same ∥s, AH, BG; then ▱$^{\text{m}}$ ABCD = ▱$^{\text{m}}$ EFGH.



Join BE, CH,

Hyp. ∵ BC = FG,
Prop. 33. EH = FG,
Ax. 1. ∴ BC = EH;

∵ str. lines EB, HC, join the extremities of = and ∥ str. lines, they are themselves = and ∥: that is,

Prop. 32. BE is = and ∥ to CH,
Def. ∴ EBCH is a ▱$^{\text{m}}$,
∴ ▱$^{\text{m}}$ EBCH = ▱$^{\text{m}}$ ABCD,
Prop. 34. since they are on the same base BC and between the same ∥s.

For the same reason,

▱$^{\text{m}}$ EFGH = ▱$^{\text{m}}$ EBCH,

∴ ▱^{m} ABCD = ▱^{m} EFGH. Ax. 1.

Wherefore parallelograms, &c.

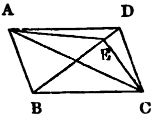
Cor. 1.—Triangles upon the equal bases and between the same parallels are equal to each other. For draw the diams. AC, EG; the △s ABC, EFG, are the halves of the equal ▱^{ms} ABCD, EFGH, and are therefore equal Ax. 7. to each other.

Cor. 2.—If a parallelogram and a triangle be upon the same or equal bases, and between the same parallels, the parallelogram is double the triangle. For the ▱^{m} ABCD is double the △ ABC, or double the △ EFG, which is its equal, by last Cor.

PROP. XXXVI. THEOR. 39. 1 Eu.

Equal triangles upon the same base, and upon the same side of it, are between the same parallels.

Let the = △s ABC, DBC, be on the same base BC, and on the same side of it; then the △s will be between the same ‖s.



Join AD,
then AD ‖ BC.

For if not,

Prop. 30. draw AE ∥ BC, and join EC.

Prop. 34, Cor. Then △ ABC = △ EBC $\left\{\begin{array}{l}\text{being on the same}\\ \text{base BC, and be-}\\ \text{tween the same ∥s}\\ \text{BC, AE ;}\end{array}\right.$

Hyp. but △ ABC = △ DBC,

Ax. 1. ∴ △ EBC = △ DBC,

i. e. less = greater,

which is impossible ;

∴ AE ∦ BC.

In the same manner it may be proved that no other line but AD ∥ BC,

∴ AD ∥ BC.

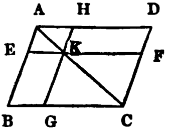
Wherefore equal triangles upon, &c.

Cor.—In the same way it may be shown, that equal triangles upon equal bases and towards the same parts, are between the same parallels.

PROP. XXXVII. THEOR. 43. 1 Eu.

The complements of the parallelograms which are about the diameter of any parallelogram, are equal to each other.

Let ABCD be a ▱$^{\text{m}}$, AC its diam. ; and EAHK, GKFC, ▱$^{\text{ms}}$ about AC, that is, through which AC passes. And BEKG, KHDF, the other ▱$^{\text{ms}}$ which make up the whole figure ABCD, are called *complements*. Then comp. BEKG = comp. KHDF.



∵ diam. AC bisects ▱^{m} ABCD,

$$\begin{array}{ll}\therefore & \left.\begin{array}{l}\triangle ABC = \triangle ADC; \\ \text{also} \left\{\begin{array}{l}\triangle AEK = \triangle AHK, \\ \triangle KGC = \triangle KFC,\end{array}\right.\end{array}\right\} \quad \text{Prop. 33.}\end{array}$$

$$\therefore \triangle AEK + \triangle KGC = \left\{\begin{array}{l}\triangle AHK + \triangle \\ KFC:\end{array}\right. \quad \text{Ax. 2.}$$

but the whole $\triangle$ ABC = whole $\triangle$ ADC;

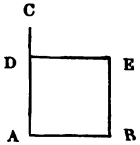
$$\therefore \left.\begin{array}{l}\text{the remain}^{g}\text{ comp.}\\ BEKG\end{array}\right\} = \left\{\begin{array}{l}\text{remain}^{g}\text{ comp.}\\ KHDF.\end{array}\right. \quad \text{Ax. 3.}$$

Wherefore the complements, &c.

PROP. XXXVIII. PROB. 46. 1 Eu.

To describe a square upon a given straight line.

Let AB be the given str. line; it is required to descr. a square upon AB.



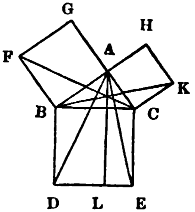
Prop. 10. Draw $AC \perp AB$,
Prop. 3. make $AD = AB$;
draw $\begin{cases} DE \parallel AB, \\ BE \parallel AD; \end{cases}$
Prop. 30.
Def. $\therefore \begin{cases} ADEB \text{ is } ▱^{m}, \\ \text{and its opp. sides are equal;} \end{cases}$
Prop. 33.
Constr. but $AB = AD$,
Ax. 1. $\therefore \begin{cases} AB = AD = DE = EB, \\ \text{and } ▱^{m} \text{ } ADEB \text{ is equilat.} \end{cases}$
Also ∵ AD meets the ∥s AB, DE,
Prop. 28. ∴ $\angle BAD + \angle ADE = 2$ rt. ∠s.
Constr. but ∠BAD is a rt. ∠;
Ax. 3. ∴ ∠ADE is a rt. ∠;
Prop. 33. but opp. ∠s of $▱^{ms}$ are equal,
Ax. 1. ∴ the ∠s ABE, BED, are rt. ∠s,
∴ the figure ADEB is rectangular and equilat.
Def. 30. ∴ the figure is a square, and descr. on AB.

Cor.—Every $▱^{m}$ that has one rt. ∠, has all its angles rt. ∠s.

PROP. XXXIX. THEOR. 47. 1 Eu.

In any right-angled triangle, the square which is described upon the hypothenuse, or side subtending the right angle, is equal to the sum of the squares described upon the sides which contain the right angle.

Let ABC be a rt. angled △, having the rt. ∠BAC. Then $BC^2 = BA^2 + AC^2$.



On BC, BA, AC, descr. the squares BE, GB, HC.[*] Prop. 38.

Draw AL || BD or CE; Prop. 30.
join AD, FC.

$\because \left\{\begin{array}{l}\angle BAC + \\ \angle BAG\end{array}\right\} = 2$ rt. $\angle$s. Hyp. & Def. 30.

$\therefore$ CA is in the same str. line with AG. Prop. 13.

For the same reason
AB is in the same str. line with AH,
and $\because \angle DBC = \angle FBA$, ea. being a rt. $\angle$. Def. 30.
Add to each $\angle ABC$;
$\therefore$ whole $\angle DBA =$ whole $\angle FBC$, Ax. 2.

$\because \left\{\begin{array}{l}AB, BD = FB, BC, \text{ ea. to ea.} \\ \angle DBA = \angle FBC,\end{array}\right.$ Def. 30.

$\therefore \triangle ABD = \triangle FBC$. Prop. 4.

Again, $▱^{m} BL = 2 \triangle ABD \left\{\begin{array}{l}\text{being on the} \\ \text{same base BD} \\ \text{and between} \\ \text{the same } \|\text{s} \\ \text{BD, AL;}\end{array}\right.$ Prop. 35. Cor. 2.

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[*] Squares and parallelograms are frequently expressed, for the sake of brevity, by the letters at their opposite angles; and the square on any line, as BC, is represented by BC^2.

and square GB = 2 △ FBC $\left\{\begin{array}{l}\text{being on the}\\ \text{same base FB}\\ \text{and between}\\ \text{the same } \|\text{s}\\ \text{FB, GC.}\end{array}\right.$

6. But the doubles of equals are equal,
∴ ▱^{m} BL = square GB.

In the same way, by joining AE, BK, it can be proved that ▱^{m} CL = square HC.

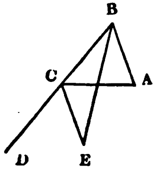
2. ∴ $\left\{\begin{array}{c}\text{whole square}\\ \text{BDEC}\end{array}\right\}$ = square GB + square HC,
i. e. $BC^2 = BA^2 + AC^2$.

Wherefore, in any right-angled triangle, &c.

PROP. XL. THEOR.

If one side of a triangle be produced, and the exterior angle, also one of the interior and opposite angles, be each of them bisected; the remaining interior and opposite angle will be double the angle made by the bisecting lines.

Let CBA be a △. Prod. BC to D, and let the ext. ∠ ACD be bisected by the str. line CE, and the int. and opp. ∠ ABC by the str. line BE. Then will ∠ BAC = 2 ∠ BEC.



1st. The lines CE, BE, will meet in some point, as E.

For $\angle ACD > \angle ABC$, Prop. 15.

$\therefore \frac{1}{2}\angle ACD > \frac{1}{2}\angle ABC$,

i. e. $\angle ECD > \angle EBC$:

add to each the $\angle$ ECB.

$\therefore \angle ECD + \angle ECB > \angle EBC + \angle ECB$;

but $\angle ECD + \angle ECB = 2$ rt. $\angle$s, Prop. 12.

$\therefore \angle EBC + \angle ECB < 2$ rt. $\angle$s.

$\therefore$ CE, BE, will meet on the same side of the str. line BC on which the $\triangle$ ABC is. Ax. 12.

$$\left.\begin{array}{l}\text{2ndly. In } \triangle \text{ BCE,}\\ \text{ext. } \angle DCE\end{array}\right\} = \left\{\begin{array}{l}\angle CBE + \angle BEC,\\ \text{the int. and opp. } \angle\text{s,}\end{array}\right.$$ Prop. 31.

$\therefore 2\angle DCE$ or $\angle DCA = 2\angle CBE + 2\angle BEC$.

but $\angle ABC = 2\angle CBE$,

$\angle DCA = \angle ABC + 2\angle BEC$.

In $\triangle$ ABC,

ext. $\angle DCA = \angle ABC + \angle BAC$; Prop. 31.

$\therefore \angle ABC + \angle BAC = \angle ABC + 2\angle BEC$. Ax. 1.

from these equals take $\angle$ ABC

$\therefore \angle BAC = 2\angle BEC$. Ax. 3.

Wherefore, if one side of a triangle, &c.

Note.—This proposition is introduced on account of the explanation that it affords of the principle upon which the sextant depends.

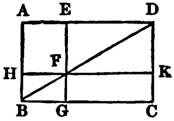
DEFINITIONS.

I.

Every right-angled parallelogram, or rectangle, is said to be contained by any two of the straight lines which contain one of the right angles.

II.

In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a *Gnomon.* 'Thus the parallelogram HG, together with the complements AF, FC is the Gnomon; which is more briefly expressed by the letters AGK or EHC, which are at the opposite angles of the parallelograms which make the gnomon.'

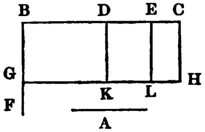


N.B.—The rectangle contained by any two straight lines AB, AD, is generally expressed by the words "rectangle of AB and AD;" or by "AB . AD."

PROP. XLI. THEOR. 1. 2 Eu.

If there be two straight lines, one of which is divided into any number of parts; the rect *angle contained by the two straight lines, is equal to the rectangles contained by the undivided line, and the several parts of the divided line.*

Let A and BC be two str. lines, and let BC be divided into any parts in D and E; then $A.BC = A.BD + A.DE + A.EC$



Draw $BF \perp BC$, Prop. 10.

make $BG = A$; Prop. 3.

draw $GH \parallel BC$; and $DK, EL, CH \parallel BG$. Prop. 30.

$\therefore A = BG = DK = EL$. Prop. 33.

$$\text{Since, rect. } BH = \begin{cases} \text{rect. } BK + \text{rect. } DL + \\ \quad \text{rect. } EH, \end{cases}$$

$$\text{And } \begin{cases} \text{rect. } BH = BG.BC = A.BC, \\ \text{rect. } BK = BG.BD = A.BD, \\ \text{rect. } DL = DK.DE = A.DE, \\ \text{rect. } EH = EL.EC = A.EC; \end{cases}$$

$$\therefore \quad A.BC = A.BD + A.DE + A.EC.$$

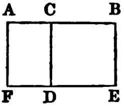
Wherefore, if there be two str. lines, &c.

PROP. XLII. THEOR. 3. 2 Eu.

If a straight line be divided into any two parts, the rectangle contained by the whole *and one of the parts,* is equal to the rect-

angle contained by the two parts, together with the square of the aforesaid part.

Let the str. line AB be divided into two parts in the pt. C; then AB . BC = AC . BC + BC^2.



Prop. 38. On BC descr. sq. CE.
prod. ED to F.
Prop. 30. draw AF ∥ CD or BE
Prop. 33 ∴ AF = CD = BE = BC.
and Def. 30. Since, rect. AE = rect. AD + rect. CE,

$$\text{And}\begin{cases}\text{rect. AE} = AB \,.\, BE = AB \,.\, BC,\\ \text{rect. AD} = AC \,.\, CD = AC \,.\, BC,\\ \text{rect. CE} = BC^2;\end{cases}$$

$$\therefore \quad AB \,.\, BC = AC \,.\, BC + BC^2.$$

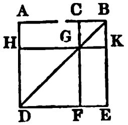
Therefore, if a str. line be, &c.

Note.—If the parts of the divided line be a and b; the proposition algebraically expressed, is $\overline{a+b} \,.\, a = ab + a^2$.

PROP. XLIII. THEOR. 4. 2 Eu.

If a straight line be divided into any two parts, the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the parts.

Let str. line AB be divided into any two parts in C, then $AB^2 = AC^2 + CB^2 + 2 . AC . CB$.



Upon AB descr. sq. ABED. Prop. 38.
join BD.
$\text{draw} \left\{ \begin{array}{l} CGF \parallel AD \text{ or } BE, \\ HGK \parallel AB \text{ or } DE; \end{array} \right.$
$\because$ BD meets $\parallel$s AD, CF,
$\therefore$ ext. $\angle BGC =$ int. oppo. $\angle ADB$; Prop. 28.
$\because$ $BA = AD$,
$\therefore$ $\angle ABD = \angle ADB$; Prop. 5.
i. e. $\angle CBG = \angle BGC$, Ax. 1.
$\therefore$ $BC = CG$, Prop. 6.
but $BC = GK$, Prop. 33.
$\therefore$ $CG = GK$; Ax. 1.
$\therefore$ $BC = CG = GK = BK$,
or fig. CK is *equilat.*
Again $\because$ CB meets $\parallel$s CG, BK,
$\therefore$ $\angle KBC + \angle GCB = 2$ rt. $\angle$s; Prop. 28.
but $\angle KBC =$ rt. $\angle$
$\therefore$ $\angle GCB =$ rt. $\angle$
$\therefore$ opp. $\angle$s CGK, GKB are rt. $\angle$s. Prop. 33.
$\therefore$ fig. CK is *rectang.*
$\therefore$ fig. $CK = CB^2$.
In the same way it may be proved, that
fig $HF = AC^2$.
Also fig. AG = fig. GE; Prop. 37.
i. e. *fig.* $AG = AC.CG = AC.CB$.

Ax. 1. $\therefore$ fig. GE = AC.CB

$\therefore$ fig. AG + fig. GE = 2 AC.CB;

$$\therefore \left\{\begin{array}{l}\text{the 4 figs. HF, CK,}\\ \text{AG, GE, i. e. the}\\ \text{whole fig. AE, or}\\ AB^2.\end{array}\right\} = \left\{\begin{array}{l} AC^2 + CB^2 + \\ 2.\,AC\,.\,CB.\end{array}\right.$$

Wherefore, if a str. line, &c.

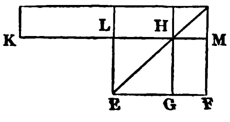
Cor. The ▱^{ms} about the dia. of a sq. are likewise squares.

Note.—If the parts of the divided line be a and b, the proposition algebraically expressed is $(a+b)^2=a^2+2ab+b^2$.

PROP. XLIV. THEOR. 5. 2 Eu.

If a straight line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equal to the square of half the whole line.

Let the str. line AB be divided into two equal parts at the pt. C, and into two unequal parts at the point D; then AD . DB + CD^2 = CB^2.



On CB descr. sq. CEFB, Prop. 38.
join BE;
draw $\left\{\begin{array}{l}\text{AK, DHG} \parallel \text{CE or BF} \\ \text{KLHM} \parallel \text{AB.}\end{array}\right.$ Prop 30.
Since, comp. CH = comp. HF Prop. 37.
add to each the fig. DM.
∴ fig. CM = fig. DF,
but, fig. CM $=\left\{\begin{array}{l}\text{fig. AL, for AC} = \\ \text{CB;}\end{array}\right.$ Prop. 35.
∴ fig. AL = fig. DF,
add to each the fig. CH,
∴ whole fig. AH = fig. DF + fig. CH,
but fig. AH $=\left\{\begin{array}{l}\text{AD . DB, for DH} \\ = \text{DB;}\end{array}\right.$ Cor. Prop. 43.
and fig. DF + fig. CH = gnomon CMG,
∴ gnomon CMG = AD . DB,
add to each the fig. LG, which is CD^2. Cor. Prop. 43.
∴ gnomon CMG + fig. LG = AD . DB + CD^2,
i. e. whole fig. CEFB or $\text{CB}^2 = \text{AD.DB} + \text{CD}^2$.

Wherefore, if a str. line, &c.

Cor.—Since $\text{CB}^2 - \text{CD}^2 = \text{AD.DB}$; the diff. of the Squares of two unequal lines is equal to the rectangle of their sum and diff.

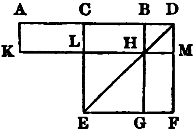
Note.—If the unequal parts of the divided line be b and c, and the half line be a; the proposition algebraically expressed is $bc + (a - b)^2 = a^2$; or $bc + (c - a)^2 = a^2$.

PROP. XLV. THEOR. 6. 2 Eu.

If a straight line be bisected, and produced to
E

any point; the rectangle contained by the whole line thus produced, and the part of it produced, together with the square of half the line bisected, is equal to the square of the straight line which is made up of the half line and the part produced.

Let the str. line AB be bisected in C, and produced to D; then $AD \cdot DB + CB^2 = CD^2$.



Prop. 38. Upon CD descr. sq. CEFD
join DE.

Prop. 29.

$$\text{Draw}\begin{cases} BHG \parallel CE \text{ or } DF,\\ KLHM \parallel AD \text{ or } EF,\\ AK \parallel CL \text{ or } DM;\end{cases}$$

Hyp. $\because$ AC = CB,
Prop. 35. $\therefore$ fig. AL = fig. CH;
Prop. 37. but, fig. CH = fig. HF,
$\therefore$ fig. AL = fig. HF,
add to each, the fig. CM.
$\therefore$ whole fig. AM = gnomon CMG,

Cor. Prop. 43. but fig. $AM = \begin{cases} AD.DB, \text{ for } DM \\ = DB; \end{cases}$

$\therefore$ gnomon CMG = AD . DB,
add to each, the fig. LG, which is equal to CB^2.
$\therefore$ gnomon CMG + fig. LG = $AD.DB + CB^2$,
i. e. whole fig. CEFD or $CD^2 = AD.DB + CB^2$.

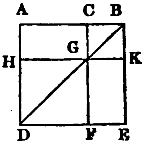
Wherefore, if a str. line, &c.

Note.—If the half line be *a*, and the part produced *b*, the proposition algebraically expressed is $(2a+b)b+a^2=(a+b)^2$.

PROP. XLVI. THEOR. 7. 2 Eu.

If a straight line be divided into any two parts, the squares of the whole line, and of one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.

Let the str. line AB be divided into any two parts at the pt. C; then $AB^2+BC^2=2.AB.BC + AC^2$.



Upon AB descr. sq. ADEB, Prop. 38.
join BD.
Draw { CGF ∥ AD or BE, Prop. 30.
HGK ∥ AB or DE ;
Since fig. AG = fig. GE, Prop. 37.
add to each the fig. CK.
∴ fig. AK = fig. CE ;
∴ fig. AK + fig. CE = 2 fig. AK
but, fig. AK + fig. CE = { gnomon AKF + fig. CK,
∴ gnomon AKF + fig. CK = 2 . fig. AK,
but, 2 AB . BC = { 2 fig. AK, for Cor. Prop.
BK = BC ;

$\therefore$ gnomon AKF + fig. CK = 2 . AB . BC,
add to each the fig. HF, which is equal to AC^2,

$$\therefore \left.\begin{array}{r}\text{gnomon AKF + fig.}\\ \text{CK + fig. HF}\end{array}\right\}=2\,.\,\text{AB}\,.\,\text{BC}+\text{AC}^2,$$

$$\text{but}\left.\begin{array}{r}\text{gnomon AKF + fig.}\\ \text{CK + fig. HF}\end{array}\right\}=\left\{\begin{array}{l}\text{whole fig. ADEB}\\ \quad\text{+ fig. CK,}\end{array}\right.$$

$$\therefore \left.\begin{array}{r}\text{whole fig. ADEB +}\\ \text{fig. CK}\end{array}\right\}=2\,\text{AB}\,.\,\text{BC}+\text{AC}^2,$$

$$\text{i.e.}\quad \text{AB}^2+\text{BC}^2=2\,\text{AB}\,.\,\text{BC}+\text{AC}^2.$$

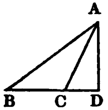
Wherefore, if a str. line, &c.

Note.—If the parts of the line be a and b, the proposition algebraically expressed is $(a+b)^2+a^2=2a(a+b)+b^2$. Or $(a+b)^2+b^2=2b(a+b)+a^2$.

PROP. XLVII. THEOR. 12. 2 Eu.

In obtuse angled triangles, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square of the side subtending the obtuse angle, is greater than the squares of the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.

Let $\triangle$ ABC have the obtuse $\angle$ ACB. From A draw AD $\perp$ BC produced: then $BA^2 > BC^2 + CA^2$ by 2 BC . CD.



∵ str. line BD is divided into 2 parts in pt. C,

∴ $BD^2 = BC^2 + CD^2 + 2 . BC . CD$, Prop. 43.

add to each DA^2

∴ $BD^2 + DA^2 = \left\{\begin{array}{l} BC^2 + CD^2 + DA^2 + \\ \quad 2\, BC . CD; \end{array}\right.$

but $\left\{\begin{array}{l} BD^2 + DA^2 = BA^2, \\ CD^2 + DA^2 = CA^2, \end{array}\right.$ Prop. 39.

∴ $BA^2 = BC^2 + CA^2 + 2\, BC . CD$,

i. e. $BA^2 > BC^2 + CA^2$ by $2\, BC . CD$.

Therefore, in obtuse angled △s, &c.

PROP. XLVIII. THEOR. 13. 2 Eu.

In every triangle, the square of the side subtending either of the acute angles, is less than the squares of the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall upon it from the opposite angle, and the acute angle.

Let △ ABC have the acute ∠ ABC, upon BC one of the sides containing it, draw AD ⊥ BC; then $AC^2 < CB^2 + BA^2$ by $2\, CB . BD$.

Fig. 1.

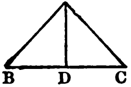


Fig. 2.

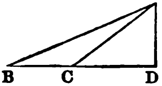
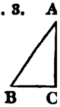


Fig. 3.



1st. When AD falls *within* the △ ABC (Fig. 1.)

$\because$ str. line CB is divided into 2 parts at D,

Prop. 46. $\therefore \quad CB^2 + BD^2 = 2 . CB . BD + DC^2$

add AD^2 to each

$\therefore CB^2 + BD^2 + AD^2 = 2.CB.BD + DC^2 + AD^2$

Prop. 39. $\text{but} \begin{cases} BD^2 + AD^2 = BA^2, \\ DC^2 + AD^2 = AC^2, \end{cases}$

$\therefore \quad CB^2 + BA^2 = 2\ CB . BD + AC^2$

i. e. $AC^2 < CB^2 + BA^2$ by $2\,CB.BD$.

2ndly. When AD falls *without* the △ ABC. (Fig. 2.)

Prop. 15. $\because \angle D = \text{rt. } \angle, \therefore \angle ACB > \text{rt } \angle$

Prop. 47. $\therefore \quad AB^2 = AC^2 + CB^2 + 2\ CB . CD,$

add CB^2 to each,

$\therefore AB^2 + CB^2 = AC^2 + 2CB^2 + 2CB . CD.$

$\because$ BD is divided into 2 parts at C.

Prop. 42. $\therefore \quad BD . CB = CB^2 + CB . CD,$

Ax. 6. $\therefore \quad 2\,BD . CB = 2\,CB^2 + 2 . CB . CD,$

$\therefore AB^2 + CB^2 = AC^2 + 2\ BD . CB,$

i. e. $AC^2 < CB^2 + AB^2$ by $2\,BD . CB$.

Lastly. When AC ⊥ BC, then is BC the str. line between the perpendicular and the acute ∠ at B. (Fig. 3.)

Prop. 39. And $AB^2 = AC^2 + CB^2,$

add CB^2 to each.

$\therefore AB^2 + CB^2 = AC^2 + 2\ CB^2,$

i. e. $AC^2 < AB^2 + CB^2$ by $2\ CB . CB$.

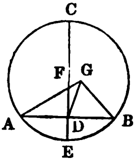
Therefore in every △, &c.

DEF.—A straight line is said to touch a circle when it meets the circle, and being produced does not cut it. This line is said to be a tangent to the circle at the point of contact.

PROP. XLIX. PROB. 1. 3 Eu.

To find the centre of a given circle.

Let ABC be a given ⊙; it is required to find its cr.



In the ⊙ draw the str. line AB, and bisect it
in D;
draw DC ⊥ AB,
prod. CD to E in ⊙,
bisect CE in F:
Then F is the Cr. of ⊙ ACB.

If not, let if possible G be the Cr.
Join GA, GD, GB.
In △s ADG, BDG,

$$\because \begin{cases} DA = DB, & \text{Const.} \\ GA = GB, & \text{15 def.} \\ DG \text{ common,} & \end{cases}$$

Prop. 7. ∴ ∠ ADG = ∠ GDB.
And ∵ they are adj. ∠s,
10 def. ∴ ∠ GDB = rt. ∠,
Const. but ∠ FDB = rt. ∠,
∴ ∠ FDB = ∠ GDB,
i. e. greater = less, which is impossible.
∴ G is not the Cr. of ⊙ ACB.

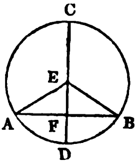
And no other point but F is the Cr.; which was to be found.

Cor.—If in a ⊙, a str. line bisect another one at rt. ∠s, the Cr. of ⊙ is in the bisecting line.

PROP. L. THEOR. 3. 3 Eu.

If a straight line drawn through the centre of a circle, bisect a straight line in it which does not pass through the centre, it shall cut it at right angles: and if it cut it at right angles, it shall bisect it.

Let ABC be a ⊙, and let CD passing through the Cr. bisect AB not passing through it, in the point F; then CD shall cut AB at rt. ∠s.



Take E the Cr. of ⊙ ABC,
join EA, EB.

1st. In △s AFE, BFE,
AF = FB, Hyp.
∵ EA = EB, 15 def.
EF common,
∴ ∠AFE = ∠BFE. Prop. 7.
And ∵ they are adj. ∠s,
∴ CD cuts AB at rt. ∠s. 10 def.

2ndly. Let CD cut AB at rt. ∠s: then shall AF=FB.

In △s AFE, BFE,
∵ EA = EB, 15 def.
∴ ∠EAF = ∠EBF; Prop. 5.
Also rt. ∠AFE = rt. ∠BFE,
and EF common,
∴ AF = FB. Prop. 25.
i. e. CD bisects AB, in pt. F.

Wherefore, if a str. line, &c.

PROP. LI. THEOR. 16. 3 Eu.

A straight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; and no straight line can be drawn from the extremity between that straight line and the circumference, so as not to cut the circle.

1st. Let ABC be a ⊙, with Cr. D and diam. AB: then the str. line drawn at rt. ∠s to AB from its extremity A, shall fall without the ⊙.

Fig. 1.

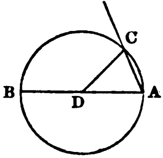


Fig 2.

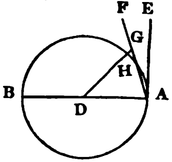


Fig. 1. If not, let it fall if possible within ⊙ as AC.

Draw DC to the point C, where AC meets the ⊙^{ce};

$\because \quad DA = DC,$

15 def. $\therefore \quad \angle DAC = \angle ACD,$

Prop. 5. but $\angle DAC = \text{rt. } \angle,$

Hyp. $\therefore \quad \angle ACD = \text{rt. } \angle,$

Prop. 16. $\therefore \quad \angle DAC + \angle ACD = \left\{\begin{array}{l} 2 \text{ rt. } \angle\text{s, which is} \\ \text{impossible.} \end{array}\right.$

∴ if AC ⊥ AB, AC does not fall *within* the ⊙. In the same manner it may be shown, that AC does not fall *upon* the ⊙^{ce}, ∴ falls *without* the ⊙ as AE, fig. 2.

Fig. 2. 2ndly. Between AE and the ⊙^{ce}, no str. line can be drawn from A, which does not cut the ⊙.

For if possible, let AF be between them.

Draw DG ⊥ AF meeting the ⊙^{ce} in H;

Const. $\because \quad \angle AGD = \text{rt. } \angle,$

Prop. 16. and $\angle DAG < \text{rt. } \angle,$

Prop. 18. $\therefore \quad DA > DG,$

but DA = DH, 15 def.
∴ DH > DG;
i. e. less > greater, which is impossible.

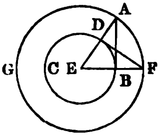
∴ no str. line can be drawn from A between AE and the $\odot^{ce}$ which does not cut the $\odot$.

Cor.—A str. line drawn perpendicular to the diam. of a $\odot$ from the $\odot^{ce}$ at the extremity of the diam.; touches the $\odot$ in one pt. only, and being produced does not cut it, and is therefore a tangent to the circle at that extremity. Def. pa. 67.

PROP. LII. PROB. 17. 3 Eu.

To draw a tangent to a given circle, from a given point, either without or in the circumference of the circle.

1st. Let A be a given point *without* the given $\odot$ BCD; it is required to draw a str. line from A, which shall touch the $\odot$.



Find E the Cr. of the $\odot$, Prop. 49.
join AE meeting $\odot$ in D.
From Cr. E at dist. EA, descr. $\odot$ AFG,
draw DF $\perp$ EA, also EBF and AB;
then AB shall touch $\odot$ BCD.

∵ E is the Cr. of ⊙s BCD, AFG.

15 def. ∴ In △s AEB, FED,
AE, EB = FE, ED, ea. to ea.
∠ at E common,

Prop. 4. ∴ ∠EBA = ∠EDF,

Const. but ∠EDF = rt. ∠,

∴ ∠EBA = rt. ∠;

and EB is drawn from the Cr. and AB ⊥ EB,

Cor. Prop. 51. ∴ AB touches the ⊙ and is drawn from A;
which was to be done.

2ndly. If the given point be *in* the $\odot^{ce}$ of the ⊙, as the point D.

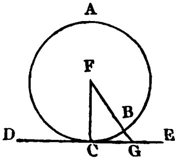
Draw DE to the Cr. E,
DF ⊥ DE,

Cor. Prop. 51. ∴ DF touches the ⊙.

PROP. LIII. THEOR. 18. 3 Eu.

If a straight line touch a circle; the straight line drawn from the centre to the point of contact, shall be perpendicular to the line touching the circle.

Let str. line DE touch ⊙ ABC in C; take the centre F, and draw str. line FC; then shall FC ⊥ DE.



For if FC be not $\perp$ DE,
draw FBG $\perp$ DE;
then $\because \angle FGC =$ rt. $\angle$,
$\therefore \angle GCF <$ rt. $\angle$, Prop. 16.
$\therefore$ $FC > FG$, Prop. 18.
but $FC = FB$, 15 def.
$\therefore$ $FB > FG$;
i. e. less > greater, which is impossible,
$\therefore$ FG is not $\perp$ DE.

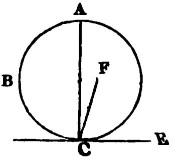
In the like manner it may be shown, that no other line but FC is $\perp$ DE,
i. e. FC $\perp$ DE.

Therefore if a str. line, &c.

PROP. LIV. THEOR. 19. 3 Eu.

If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

Let str. line DE touch ⊙ ABC in C; draw CA $\perp$ DE: then the Cr. of ⊙ shall be in CA.



For if not, if possible let F be the Cent. of ⊙, join CF;

Then ∵ DE touches ⊙ ABC,
and FC is drawn from the Cr. to point of contact,
Prop. 53. ∴ FC ⊥ DE,
∴ ∠FCE = rt. ∠,
Hyp. but ∠ACE = rt. ∠,
∴ ∠FCE = ∠ACE,
i. e. less = greater, which is impossible,
∴ F is not Cr. of ⊙ ABC.

In like manner it may be shown that no other pt. which is not in CA, is the Cr. of ⊙ ABC;

i. e. the Cr. of ⊙, is in CA.

Therefore, if a str. line, &c.

PROP. LV. THEOR. 20. 3 Eu.

The angle at the centre of a circle is double of the angle at the circumference upon the same arc, that is, upon the same part of the circumference.

Let ABC be a ⊙; BEC an ∠ at the Cr. E, and BAC an ∠ at the ⊙^{ce}, having the same arc BC for their base: then shall ∠BEC = 2∠BAC.

Fig. 1.

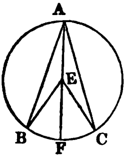
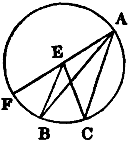


Fig. 2.



1st. Let Cr. E be *within* the ∠ BAC, Fig. 1.
join AE and prod. it to F in the ⊙^{ce}:
∵ EA = EB, 15 def.
∴ ∠ EAB = ∠ EBA, Prop. 5.
∴ ∠ EAB + ∠ EBA = 2 ∠ EAB;
but ∠ EAB + ∠ EBA = ext. ∠ BEF, Prop. 31.
∴ ∠ BEF = 2 ∠ EAB:
Similarly ∠ FEC = 2 ∠ EAC,
∴ whole ∠ BEC = 2 . whole ∠ BAC.

2ndly. Let Cr. E be *without* the ∠ BAC. Fig. 2.
It may be demonstrated as before, that
∠ FEC = 2 ∠ FAC,
∠ FEB = 2 ∠ FAB,
∴ rem. ∠ BEC = 2 . rem. ∠ BAC.

Therefore, the angle at the centre, &c.

PROP. LVI. THEOR. 21. 3 Eu.

The angles in the same segment of a circle are equal to one another.

Let ABCD be a ⊙, and ∠s BAD, BED in the same segment BAED: then shall ∠ BAD = ∠ BED.

Fig. 1.

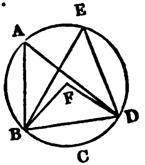
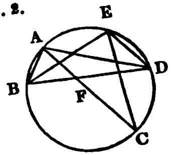


Fig. 2.



Let F be Cr. of ⊙ ABCD.

1st Case. Let seg. BAED > semi ⊙.

Join BF, FD:

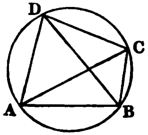
Fig. 1. Then ∵ { ∠BFD is at Cr.,
∠BAD is at ⊙^{ce}.
And both on the same part BCD of the ⊙^{ce},
∴ ∠BFD = 2∠BAD,
Prop. 55. Similarly ∠BFD = 2∠BED;
∴ ∠BAD = ∠BED.

7 Ax.
Fig. 2. 2nd Case. Let seg. BAED be not > semi ⊙.
Draw AF to Cr. and prod. it to ⊙^{ce} at C,
Join CE and BD:
∴ seg. BADC > semi ⊙.
And ∠s in it are equal by the 1st Case:
∴ ∠BAC = ∠BEC.
Similarly ∠CAD = ∠CED,
∴ whole ∠BAD = whole ∠BED.
Wherefore, the angles, &c.

PROP. LVII. THEOR. 22.

The opposite angles of any quadrilat figure inscribed in a circle, are toge equal to two right angles.

Let ABCD be a quadrilat. fig. in ⊙ ABCD: then any two of its opp. ∠s shall be together equal to 2 rt. ∠s.



Join AC, BD.

$$\text{Then} \because \left\{ \begin{array}{l} \angle CAB = \angle CDB, \\ \angle BCA = \angle ADB, \end{array} \right.$$ Prop. 56.

$$\therefore \quad \angle CAB + \angle BCA = \text{whole } \angle ADC;$$

to each add $\angle ABC$,

$$\therefore \angle ABC + \left. \begin{array}{r} \angle CAB + \\ \angle BCA \end{array} \right\} = \angle ABC + \angle ADC,$$

$$\text{but } \angle ABC + \left. \begin{array}{r} \angle CAB + \\ \angle BCA \end{array} \right\} = 2 \text{ rt. } \angle s,$$ Prop. 31.

$$\therefore \quad \angle ABC + \angle ADC = 2 \text{ rt. } \angle s.$$

In like manner it may be shown that

$$\angle BAD + \angle DCB = 2 \text{ rt. } \angle s.$$

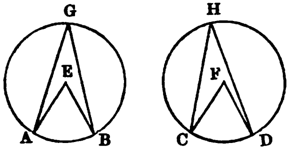
Therefore, the opp. ∠s, &c.

PROP. LVIII. THEOR. 26. 3 Eu.

In the same or equal circles, equal angles at the centre or the circumference stand upon equal arcs.

Let ABG, CDH be equal ⊙s, and the *equal angles* AEB, CFD at their centres, and

AGB, CHD at their circumferences; then shall arc AB = arc CD.



For, let str. line AE be applied to str. line CF, so that pt. A may coincide with pt. C, and AE fall upon CF.

Then ∵ AE = CF
∴ pt. E coincides with pt. F.
And ∵ { AE coincides with CF,
∠AEB = ∠CFD,
∴ EB falls upon FD.
And ∵ EB = FD
∴ pt. B coincides with pt. D;

Also ∵ every point in arc AB is the same dist. from E, as every point in arc CD, is from F; and the points A, B, coincide with points C, D,

∴ arc AB coincides with arc CD,
i. e. arc AB = arc CD;

∴ the equal ∠s at the Cr. AEB, CFD, as also the ∠s at the $\odot^{ce}$ AGB, CHD which are also equal (Ax. 7) stand upon equal arcs AB and CD.

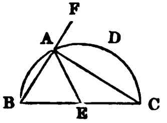
Wherefore, in the same, &c.

PROP. LIX. THEOR. 31. 3 Eu.

The angle in a semicircle is a right angle.

Let BADC be a semi ⊙, of which the diam. is BC and centre E; also BAC an ∠ contained in the semi ⊙.

Then ∠ BAC = rt. ∠

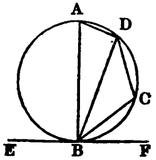


Join EA and prod. BA to F.
Then, ∵ BE = EA,
∴ ∠ EAB = ∠ ABC. Prop. 5.
And ∵ AE = EC,
∴ ∠ EAC = ∠ ACB, Prop. 5.
∴ whole ∠ BAC = ∠ ABC + ∠ ACB;
but, ext. ∠ FAC = ∠ ABC + ∠ ACB, Prop. 31.
∴ ∠ BAC = ∠ FAC,
∴ each of the ∠s BAC, FAC, is a rt. ∠, 10 def.
∴ ∠ BAC in the semi ⊙ = rt. ∠.
Wherefore the ∠, &c.

PROP. LX. THEOR. 32. 3 Eu.

If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angles which this line makes with the line touching the circle, shall be equal to the angles which are in the alternate segments of the circle.

Let str. line EF touch ⊙ ABCD in B; from B draw BD, cutting the ⊙: then the ∠s which BD makes with the touching line E shall be equal to the $\angle$s in the alternate segments of the $\odot$; that is, $\angle$DBF = $\angle$ i seg. DAB; and $\angle$DBE = $\angle$ in seg. DCB.



Draw BA $\perp$ EF.
Take any point C in $\odot^{ce}$ DB;
join AD, DC, CB.

Then $\because$ EF touches $\odot$ in B,
and BA $\perp$ EF at the point of contact B;
Prop. 54. $\therefore$ Cr. of $\odot$ is in str. line BA;
Prop. 59. $\therefore$ $\angle$ADB in semi $\odot$ = rt. $\angle$,
Prop. 31. $\therefore$ $\angle$BAD + $\angle$ABD = rt. $\angle$;
Const. but $\angle$ABF = rt. $\angle$,
$\therefore$ $\angle$ABF = $\angle$BAD + $\angle$ABD;
take away the common $\angle$ABD,
$\therefore$ rem. $\angle$DBF = $\left\{\begin{array}{l}\text{rem. } \angle\text{BAD, in a}\\ \text{tern. seg. DAB.}\end{array}\right.$

Again $\because$ ABCD is a quadrilat. fig. in a $\odot$;
Prop. 57. $\therefore$ $\angle$BAD + $\angle$BCD = 2 rt. $\angle$s,
Prop. 12. but $\angle$DBF + $\angle$DBE = 2 rt. $\angle$s,
$\therefore$ $\angle$DBF + $\angle$DBE = $\angle$BAD + $\angle$BCD;
and $\left.\begin{array}{l}\text{it has been}\\ \text{proved } \angle\text{DBF}\end{array}\right\}$ = $\angle$BAD,

$\therefore$ rem. $\angle$DBE $= \begin{cases} \text{rem. } \angle \text{BCD, in the} \\ \text{altern. seg. DCB.} \end{cases}$

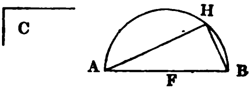
Wherefore if a str. line, &c.

PROP. LXI. PROB. 33. 3 Eu.

Upon a given straight line to describe a segment of a circle, which shall contain an angle equal to a given rectilineal angle.

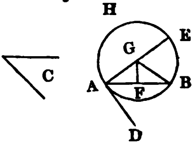
Let AB be a given str. line, C the given ∠; it is required to descr. on AB, a seg. of ⊙, that shall contain an ∠ = ∠ C.

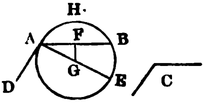
1st. If ∠ C be a rt. ∠.



Bisect AB in F.
From Cr. F, and dist. FB, descr. semi ⊙ AHB;
Then, any ∠ AHB in semi ⊙ = ∠ C. Prop. 59.

2ndly. If ∠ C be not a rt. ∠.





Make ∠ BAD = given ∠ C. Prop. 22.
Bisect AB in F,

draw $\left\{\begin{matrix} AE \perp AD \\ FG \perp AB \end{matrix}\right\}$ join GB.

$\because \left\{\begin{matrix} AF, FG = BF, FG \text{ ea. to ea.} \\ \angle AFG = \angle BFG, \end{matrix}\right.$

Prop. 4. $\therefore$ $AG = GB$,

$\therefore$ A ⊙ descr. from Cr. G, dist. GA, passes through B.

Let this ⊙ be AHB.

Then $\because$ AD $\perp$ AE at the extrem. of diam. AE,

Cor. Prop. 51. $\therefore$ AD touches the ⊙;

And $\because$ AB from pt. of contact, cuts ⊙,

Prop. 60. $\therefore$ $\angle DAB = \angle$ in alt. seg. AHB,

Const. but $\angle DAB = \angle C$,

$\therefore$ $\angle C = \angle$ in alt. seg. AHB

Wherefore, upon the line AB, the seg. AHB of a ⊙ is described which contains an $\angle =$ given $\angle C$.

PROP. LXII. THEOR. 35. 3 Eu.

If two straight lines within a circle cut one another, the rectangle contained by the segments of one of them, is equal to the rectangle contained by the segments of the other.

Let the two str. lines AC, BD cut each other in E within ⊙ ABCD: then AE . EC = BE . ED.

Fig. 1.

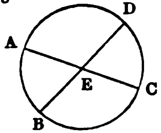


Fig. 2.

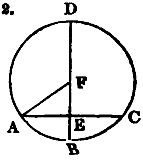


Fig. 3.

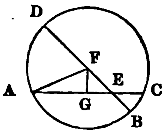
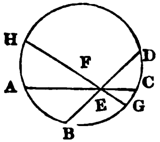


Fig. 4.



1st. Let AC, BD pass each through the Cr. E. Fig. 1.

$\because AE = EC = BE = ED,$ 15 def.

$\therefore AE \,.\, EC = BE \,.\, ED.$

2ndly. Let one of them BD pass through the Cr., and cut the other AC which does not pass through the Cr. at rt. ∠s in the point E. Fig. 2.

Bisect BD in F,

∴ F is cent. of ⊙ ABCD; Prop. 49.

join AF:

∵ BD which passes through the Cr. cuts AC which does not pass through the Cr. at rt. ∠s in E,

$\therefore AE = EC,$ Prop. 50.

∵ BD is cut into equal parts in F, and in 2 unequal parts in E,

$\therefore BE \,.\, ED + EF^2 = FB^2$ or FA^2, Prop. 44.

but $AE^2 + EF^2 = FA^2$, Prop. 39.

$\therefore BE \,.\, ED + EF^2 = AE^2 + EF^2,$

taking away EF^2 which is common,

$\therefore$ $BE \,.\, ED = AE^2$ or $AE \,.\, EC.$

3rdly. Let BD which passes through the Cr. cut AC which does not pass through the Cr. in E, but not at rt. ∠s. Fig. 3.

Then as before if BD be bisected in F, F is Cr. of ⊙.

Join AF, draw $FG \perp AC$;

Prop. 50. $\therefore \quad AG = GC,$

Prop. 44. $\therefore AE . EC + EG^2 = AG^2,$

add GF^2 to each,

$\therefore AE . EC + EG^2 + GF^2 = AG^2 + GF^2;$

Prop. 39. $\text{but} \left\{ \begin{array}{l} EG^2 + GF^2 = EF^2, \\ AG^2 + GF^2 = AF^2, \end{array} \right.$

$\therefore AE . EC + EF^2 = AF^2$ or FB^2,

Prop. 44. but $FB^2 = BE . ED + EF^2,$

$\therefore AE . EC + EF^2 = BE . ED + EF^2.$

Take away EF^2 which is common.

$\therefore \quad AE . EC = BE . ED.$

Fig. 4. Lastly. Let neither AC nor BD pass through the Cr.

Take the Cr. F, through E the intersection of AC and DB, draw the diam. GEFH;

$\text{By 3rd Case} \left\{ \begin{array}{l} AE . EC = GE . EH, \\ BE . ED = GE . EH, \end{array} \right.$

$\therefore \quad AE . EC = BE . ED.$

Wherefore, if 2 str. lines, &c.

PROP. LXIII. THEOR. 36. 3 Eu.

If from any point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square of the line which touches it.

Let D be any point without the $\odot$ ABC; and DCA, DB two str. lines drawn from it, of which DCA cuts the $\odot$, and DB touches it: then $AD . DC = DB^2$. (Figs. 1 and 2.)

Fig. 1.

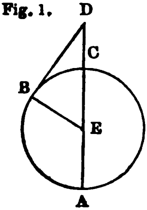


Fig. 2.

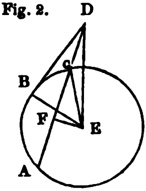
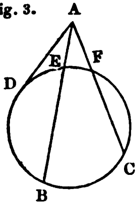


Fig. 3.



1st. Let DCA pass through the Cr. E. Fig. 1.

join EB;

$\therefore \angle EBD =$ rt. $\angle$ Prop. 53.

And $\because$ AC is bisected in E and prod. to D,

$\therefore AD . DC + EC^2 = ED^2$, Prop. 45.

but $EC = EB$,

$\therefore AD . DC + EB^2 = ED^2$;

And since $\angle EBD =$ rt. $\angle$,

$\therefore ED^2 = EB^2 + DB^2$, Prop. 39.

$\therefore AD . DC + EB^2 = EB^2 + DB^2$

take away EB^2 which is common,

$\therefore AD . DC = DB^2$.

2ndly. If DCA do not pass through the Cr. Fig. 2.

Take E the Cr., draw EF $\perp$ AC;

join EB, EC, ED:

$\because$ EF which passes through the Cr. cuts AC which does not pass through the Cr. at rt. $\angle$s, it also bisects it:

$\therefore AF = FC$; Prop. 50.

And $\because$ AC is bisected in F, and prod. to D,

Prop. 45. $\therefore \quad AD \cdot DC + FC^2 = FD^2$:
to each add FE^2,
$\therefore \ AD \cdot DC + FC^2 + FE^2 = FD^2 + FE^2$,
and since $\angle EFD = \text{rt. } \angle$,
Prop. 39. $\therefore \quad \begin{cases} ED^2 = FD^2 + FE^2, \\ EC^2 = FC^2 + FE^2, \end{cases}$
$\therefore \ AD \cdot DC + EC^2 = ED^2$;
but $EC = EB$,
$\therefore \ AD \cdot DC + EB^2 = ED^2$;
and since $\angle EBD = \text{rt. } \angle$,
Prop. 39. $\therefore \quad EB^2 + DB^2 = ED^2$,
$\therefore \ AD \cdot DC + EB^2 = EB^2 + DB^2$,
take away EB^2 which is common,
$\therefore \quad AD \cdot DC = DB^2$.

Wherefore, if from any point, &c.

Cor.—If from any pt. without a $\odot$, str. lines as AB, AC be drawn (fig. 3); the rectangles contained by the whole lines and the parts of them without the $\odot$, are equal to one another, viz.:

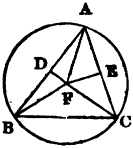
$$BA \cdot AE = CA \cdot AF;$$

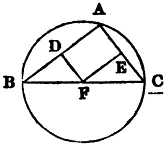
for each of these rectangles is equal AD^2.

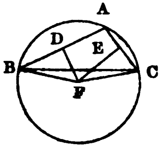
PROP. LXIV. PROB. 5. 4 Eu.

To describe a circle about a given triangle.

Let the given $\triangle$ be ABC; it is required to describe a $\odot$ about the $\triangle$ ABC, so that the $\odot$ may pass through the angular points A, B, C of the $\triangle$.







Bisect AB and AC in D and E.

$$\text{Draw}\begin{cases}DF \perp AB,\\ EF \perp AC,\end{cases}$$

then DF and EF prod. will meet each other;
For if not, then DF $\parallel$ EF, Def. 35.
$\therefore$ AB and AC, at rt. $\angle$s to them are $\parallel$ which is absurd,
$\therefore$ DF and EF do meet each other.
Let them meet in F,
join AF;
If F be not in BC, join BF, CF;
then in $\triangle$s DBF, ADF,

$$\because\begin{cases}AD = DB,\\ DF \text{ common},\\ \angle ADF = \angle BDF, \text{ being rt. } \angle\text{s}:\end{cases}$$

$\therefore$ base FA = base FB; Prop. 4.
similarly FC = FA,
$\therefore$ FB = FC,
$\therefore$ FA = FB = FC.
$\therefore$ A$\odot$ descr. from Cr. F, at the distance of one of them, will pass through A, B, C the angular points of the $\triangle$; which was to be done.

Def. 1.—Quantities of the same kind may be compared with each other with respect to their magnitudes, by means of the number of equal parts or units they each contain; and *ratio is the relation which two such quantities have to each other, determined by considering what fractional part one is of the other when both quantities are numerically expressed;* as one line is compared with another, by means of the number of equal units, as inches, &c., and one surface with another surface, by the number of square or superficial units which they respectively contain. If the number of equal units contained in 2 geometrical quantities of the same kind be represented by a and b, the ratio of the quantities is therefore expressed by the fraction $\frac{a}{b}$. The meaning of which is, that one quantity being divided into as many equal parts as a expresses, the other is divisible into a number of the same parts equal to b.

Geometrical quantities may also be so related to each other, as to admit of no numbers that will *strictly* represent them, as the side and diagonal of a square, which by Prop. 39, are as 1 to $\sqrt{2}$, or 1 to 1,414, &c.; since they do not each contain an *exact* number of inches, or equal parts of an inch, without a remainder; yet by the *subdivision* of the inch (or whatever unit of measurement is used) into any number of equal parts, the ratio of the lines may be obtained to any assigned degree of accuracy. By the continued subdivision of the unit, the remainder, or undivided part of any quantity may be conceived to be so diminished, as to be of no assignable value compared with the other part, and as there are no conditions which limit this subdivision, the ratio of these quantities may be expressed as before by the fraction $\frac{a}{b}$.

Def. 2.—*A proportion consists of 2 equal ratios.* Or if, of four quantities; the first contains the second, or part of the second, as often as the third contains the fourth, or like part of the fourth; they are said to be proportionals. That is, if a, b, c, d be the numerical values of 4 geometrical quantities; the proportion is represented by the equation $\frac{a}{b}=\frac{c}{d}$; which is usually expressed by saying, *a is to b as c is to d.* In this proportion, a and c are called the *antecedents*, and b and d the *consequents*. The proportion is also expressed by $a:b::c:d$, or by $a:b=c:d$ as most convenient; in which a and d are called the *extremes*, and b and c the *means*. And in any proportion if the means are equal to each other, *i. e.* if $a:b::b:c$, then b is said to be a *mean proportional* between a and b.

Def. 3.—Similar rectilineal figures are those which have their respective angles equal to each other, and the sides about the equal angles proportional.

PROP. LXV. THEO

If four geometrical quantities, represented by a, b, c, d, be p that is, if $a : b :: c : d$; then will

Hyp.
$$\text{Since } \frac{a}{b} = \frac{c}{d},$$
$$\therefore \quad ad = bc.$$

PROP. LXVI. THE

If there be four quantities repre fore by a, b, c, d; and if $ad =$ $a : b :: c : d$.

Hyp.
$$\text{Since } ad = bc,$$
$$\therefore \quad \frac{a}{b} = \frac{c}{d}.$$
That is, $a : b :: c : d$.

PROP. LXVII. THE

If as before $a : b :: c : d$; the $a : c :: b : d$.

Prop. 65.
$$\text{Since } ad = bc,$$
$$\therefore \quad \frac{a}{c} = \frac{b}{d}.$$
That is, $a : c :: b : d$.

PROP. LXVIII. THEOR. 12. 5 Eu.

If there be any number of proportional quantities, represented numerically as $a : b :: c : d :: e : f$, &c.; then will one antecedent be to its consequent, as the sum of all the antecedents to the sum of all the consequents; or $a : b :: a+c+e$, &c.: $b+d+f$, &c.

$$\text{Since } \left\{ \begin{array}{c} \frac{a}{b}=\frac{a}{b} \\ \frac{a}{b}=\frac{c}{d} \\ \frac{a}{b}=\frac{e}{f} \end{array} \right\} \therefore \quad \begin{array}{c} ab=ba, \\ ad=bc, \\ af=be. \end{array}$$

$\therefore\ ab+ad+af$, &c. $= ba+bc+be$, &c.

That is, $a(b+d+f) = b(a+c+e)$,

$\therefore\ a : b :: a+c+e$, &c. : $b+d+f$, &c. Prop. 66.

PROP. LXIX. THEOR.

If as before $a : b :: c : d$, then will $a^2 : b^2 :: c^2 : d^2$.

$$\text{Since } \frac{a}{b}=\frac{c}{d}, \qquad \text{Hyp.}$$

$$\therefore \quad \frac{a^2}{b^2}=\frac{c^2}{d^2},$$

or $a^2 : b^2 :: c^2 : d^2$.

PROP. LXX. THEOR. 15. 5 Eu.

Equimultiples of two quantities have the same ratio as the quantities themselves.

Let ma and mb be equimultiples of a and b; then will $a : b :: ma : mb$.

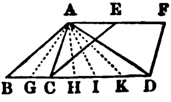
For $\frac{a}{b} = \frac{ma}{mb}$.

That is, $a : b :: ma : mb$.

PROP. LXXI. THEOR. 1. 6 Eu.

Triangles and parallelograms of the same altitude are to each other as their bases.

In str. line BD take any number of equal parts or units BG, GC, CH, &c. From pt. A without the line BD draw AB, AG, AC, AH, &c.: also, draw AF, EC, FD respectively ∥ BD, AB, AC; and the △s ABC, ACD having the same altitude, viz. the perpendicular drawn from A to BD; then will base BC : base CD :: △ ABC : △ ACD :: ▱^{m} ABCE : ▱^{m}ACDF.



op. 35. r. 1. Then △s ABG, AGC, ACH, &c. are equal to each other;

$$\text{And}\begin{cases}\text{No. of parts in BC} = \text{No. of }\triangle\text{s in ABC,}\\ \text{No. of parts in CD} = \text{No. of }\triangle\text{s in ACD,}\end{cases}$$

$$\therefore \frac{\text{No. of parts in BC}}{\text{No. of parts in CD}} = \frac{\text{No. of }\triangle\text{s in ABC}}{\text{No. of }\triangle\text{s in ACD}}$$

$$\text{i. e. } \frac{BC}{CD} = \frac{\triangle ABC}{\triangle ACD}$$

$$\text{or, Base } BC : \text{Base } CD :: \triangle\, ABC : \triangle ACD$$

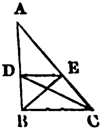
$$:: \text{▱}^{m} ABCE : \text{▱}\ ACDF. \qquad \text{Prop. 70.}$$

Therefore triangles, &c.

PROP. LXXII. THEOR.

If a straight line be drawn parallel to one of the sides of a triangle, it shall cut the other sides proportionally: and if the sides be cut proportionally, the straight line which joins the points of section shall be parallel to the remaining side of the triangle.

1st. Draw DE ∥ BC one of the sides of the △ABC; then shall DB : AD :: EC : AE; also AB : AC :: AD : AE.



Join BE, CD:

$\because$ $\triangle$s BDE, CDE, are on the same bas
and between the same ∥s DE and BC

Prop. 34. Cor. $$\therefore \triangle BDE = \triangle CDE,$$

$$\therefore \frac{\triangle BDE}{\triangle ADE} = \frac{\triangle CDE}{\triangle ADE},$$

but $\triangle$'s having the same alt., are as t

Prop. 71. bases,

$$\therefore \begin{cases} \dfrac{\triangle BDE}{\triangle ADE} = \dfrac{DB}{AD}, \\ \dfrac{\triangle CDE}{\triangle ADE} = \dfrac{EC}{AE}, \end{cases}$$

Ax. 1. $$\therefore \frac{DB}{AD} = \frac{EC}{AE};$$

i. e. DB : AD :: EC : AE.

Prop. 67. Also DB : EC :: AD : AE,

Prop. 68. $\therefore$ DB + AD : EC + AE :: AD : AE,

i. e. AB : AC :: AD : AE.

2nd. Let the sides AB, AC of the $\triangle$ be cut proportionally in pts. D, E, so DB : AD :: EC : AE, or AB : AC :: AD : join DE: then shall DE ∥ BC.

$$\text{As before} \begin{cases} \dfrac{\triangle BDE}{\triangle ADE} = \dfrac{DB}{AD}, \\ \dfrac{\triangle CDE}{\triangle ADE} = \dfrac{EC}{AE}, \end{cases}$$

Hyp. and since $$\frac{DB}{AD} = \frac{EC}{AE},$$

$$\therefore \quad \frac{\triangle BDE}{\triangle ADE} = \frac{\triangle CDE}{\triangle ADE},$$

$$\therefore \quad \triangle BDE = \triangle CDE,$$

But $\triangle$s BDE, CDE are on the same base DE, and $\therefore$ are between the same $\parallel$s;

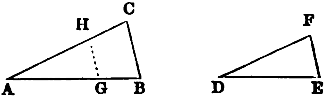
$$\therefore \quad DE \parallel BC.$$ Prop. 36.

Wherefore if a str. line, &c.

PROP. LXXIII. THEOR.

Equiangular triangles have their corresponding sides about the equal angles in the same proportion.

Let ABC, DEF be equiangular $\triangle$s, having $\angle$s A, B, C, respectively equal to $\angle$s D, E, F, ea. to ea.; then shall AB : AC :: DE : DF; or AB : DE :: AC : DF.



Conceive $\triangle$DEF so applied to $\triangle$ABC, that D may coincide with A, and DE the lesser fall upon AB the greater.

$$\text{Take} \begin{cases} AG = DE, \\ AH = DF, \end{cases}$$

Join GH.

$$\text{Then} \because \begin{cases} AG, AH = DE, DF, \text{ ea. to ea.} \\ \angle GAH = \angle EDF, \end{cases}$$

$$\therefore \quad \angle AGH = \angle DEF;$$ Prop. 4.

Hyp. but $\angle ABC = \angle DEF$

Prop. 27. $\therefore \angle AGH = \angle ABC,$

Prop. 72. $\therefore GH \parallel BC,$

$\therefore AB : AC :: AG : AH$

Prop. 67. i. e. $AB : AC :: DE : DF$

And $AB : DE :: AC : DF$

Similarly $\begin{cases} AC : BC :: DF : EF, \\ AB : BC :: DE : EF. \end{cases}$

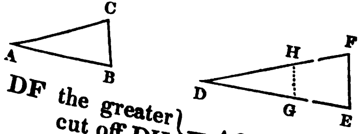
Therefore equiangular triangles,

Def. 3. pa. 89. COR.—Equiangular triangles are

PROP. LXXIV. THEOR

Two triangles are equiangular, when in the one is equal to an angle in t and the sides about the equal an proportionals.

In △s ABC, DEF, let $\angle A = \angle D$ a $AB :: DF : DE$; then will the △s be angular, and the ∠s A, B, C = ∠s D, ea. to ea.



From DF the greater} cut off DH } = AC the lesser.

And $DG = AB$.

Join HG.

In △s ABC, DGH.

Since $\begin{cases} CA, AB = HD, DG, \text{ ea. to e} \\ \angle A = \angle D, \end{cases}$

$$\therefore \begin{cases} \angle DHG = \angle C, \\ \angle DGH = \angle B, \end{cases}$$ Prop. 4.

And $\because$ AC : AB :: DF : DE;

That is DH : DG :: DF : DE; Hyp.

$\therefore$ HG ∥ FE;

$$\therefore \begin{cases} \angle DHG = \angle F, \\ \angle DGH = \angle E, \end{cases}$$ Prop. 72. Prop. 28.

$\therefore \angle B = \angle E,\ \angle C = \angle F,$ ..

i. e. $\angle$s A, B, C = $\angle$s D, E, F; ea. to ea.

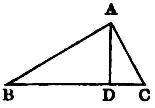
$\therefore$ △s ABC, DEF are equiangular.

Wherefore if two triangles, &c.

PROP. LXXV. THEOR. 8. 6 Eu.

In a right angled triangle, if a perpendicular be drawn from the right angle to the opposite side, the triangles on each side of the perpendicular are similar to the whole triangle, and to each other.

Let ABC be a rt. angled △, having the rt. $\angle$ BAC; draw AD ⊥ BC; then the △s ABD, ADC are similar to the whole △ ABC, and to one another.



In the △s ABC, ABD,

$$\because \begin{cases} \angle BAC = \angle ADB, \text{ ea. being a rt. } \angle, \\ \angle B \text{ common to both } \triangle\text{s}; \end{cases}$$

Prop. 31. ∴ rem. ∠ACB = rem. ∠BAD;
And △s ABC, ABD are equia
also the corresponding sides are propo
(Prop. 73); and the △'s are simila
Prop. 73).

In the same way it may be show
△ADC is equiangular and similar to
and also to △ABD.

Therefore in a rt. angled △, &c.

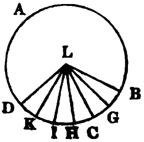
Cor. The perpendicular AD is a
proportional between the segments B
of the base BC. And each of the si
AC, is a mean proportional between t
BC and the segment adjacent to the si
For in the

Prop. 73. △s, BDA, ADC, BD : DA :: DA
.. △s, ABC, DBA, BC : BA :: BA
.. △s ABC, ACD, BC : CA :: CA

PROP. LXXVI. THEOR.

In the same or equal circles, angles at
tres have the same ratio which the
which they stand have to each other.

In the ⊙ ABD, whose centre is
any No. of = ∠s DLK, KLI, ILH, &c
shall arc DC : arc CB :: ∠DLC : ∠C



Since ∠ s DLK, KLI, &c. = each other,
∴ arcs DK, KI, &c. = each other; Prop. 58

$$\text{And} \because \begin{cases} \text{No. of arcs in DC} = \begin{cases}\text{No. of } \angle\text{s in} \\ \text{DLC,}\end{cases} \\ \text{No. of arcs in CB} = \begin{cases}\text{No. of } \angle\text{s in} \\ \text{CLB;}\end{cases} \end{cases}$$

$$\therefore \frac{\text{No. of arcs in DC}}{\text{No. of arcs in CB}} = \frac{\text{No. of } \angle\text{s in DLC}}{\text{No. of } \angle\text{s in CLB}};$$

$$\text{i. e. } \frac{\text{Arc DC}}{\text{Arc CB}} = \frac{\angle \text{DLC}}{\angle \text{CLB}}.$$

$$\text{or Arc DC} : \text{Arc CB} :: \angle \text{DLC} : \angle \text{CLB}.$$

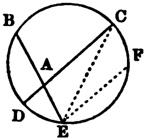
Wherefore in the same, &c.

Since angles at the centre of a circle vary as the contained arcs upon which they stand; it is frequently necessary for *practical purposes* to consider *an arc as the measure of an angle;* or the angle to be measured by the number of equal units or degrees contained in its corresponding arc, the whole circumference of the circle being usually divided into 360 degrees.

PROP. LXXVII. THEOR.

The angle made by the intersection of two straight lines within a circle is measured by half the sum of the two contained arcs.

Let str. lines BE and CD cut each other in pt. A, within the ⊙BCFD; then will half the sum of the arcs DE and BC be the measure of the ∠ of intersection BAC.



Draw EF ∥ DC;
Join CE,

.28. then $\because \begin{cases} \angle \text{DCE} = \angle \text{CEF}, \\ \text{And} = \angle\text{s at the (} \\ \text{stand on} = \text{arcs;} \end{cases}$
.58.

$\therefore$ Arc DE = arc CF.
To each add arc BC,
$\therefore$ Arc DE + arc BC = Arc BCF.

But $\angle$ BEF at the $\odot^{ce}$, or its equal $\angle$ B is measured by half the arc BCF;

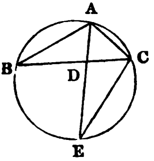
$\therefore$ $\angle$ BAC is measured by $\dfrac{\text{Arc BC + arc}}{2}$

Cor.—Errors of "Excentricity" in ins ments for the measurement of angles, are rected by taking the half sum, or mean, tween the readings of two opposite Vernie

PROP. LXXVIII. THEOR.

If an angle of a triangle be bisected b straight line which cuts the base; the angle contained by the sides of the tria is equal to the rectangle contained by segments of the base, together with square of the bisecting line.

Let ABC be a △, and the str. line AD bisect the ∠ BAC; then shall $BA \cdot AC = BD \cdot DC + AD^2$.



About △ ABC descr. ⊙ ACB, Prop. 64.
prod. AD to meet ⊙ in E,
join EC.

$$\text{Then} \because \begin{cases} \angle BAD = \angle CAE, & \text{Hyp.} \\ \angle ABD = \begin{cases} \angle AEC, \text{ being in the} \\ \text{same seg.} \end{cases} & \text{Prop. 56.} \end{cases}$$

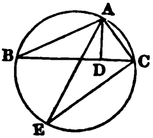
∴ △s ABD, AEC are equiangular:
∴ BA : AD :: EA : AC, Prop. 73.
∴ $BA \cdot AC = EA \cdot AD$; Prop. 65.
but $ED \cdot DA + AD^2 = EA \cdot AD$; Prop. 42.
∴ $BA \cdot AC = ED \cdot DA + AD^2$;
and $ED \cdot DA = BD \cdot DC$; Prop. 62.
∴ $BA \cdot AC = BD \cdot DC + AD^2$.

Wherefore, if an ∠, &c.

PROP. LXXIX. THEOR.

If from any angle of a triangle a straight line be drawn perpendicular to the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

Let ABC be a △, and AD ⊥ BC
the diam. of the circumscribing ⊙ A]
then shall $BA \cdot AC = AD \cdot AE$.



Join EC.

Prop. 59. Prop. 56. Then ∵ $\begin{cases} \text{rt. } \angle BDA = \angle ECA \text{ in semi} \\ \text{and } \angle ABD = \angle AEC\text{, in same} \end{cases}$

∴ △s ABD, AEC are equiangular

Prop. 73. ∴ $BA : AD :: EA : AC$;

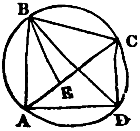
Prop. 65. ∴ $BA \cdot AC = AD \cdot EA$.

If therefore any angle, &c.

PROP. LXXX. THEOR.

The rectangle contained by the diagonals quadrilateral figure inscribed in a c is equal to both the rectangles together, tained by its opposite sides.

Let ABCD be any quadrilateral fig. ins in a ⊙; draw the diagonals AC, BD: shall $AC \cdot BD = AB \cdot CD + AD \cdot BC$.



Make $\angle ABE = \angle DBC$:

add to each $\angle EBD$;

$\therefore \angle ABE + \angle EBD = \angle DBC + \angle EBD$;

i. e. $\angle ABD = \angle EBC$:

but $\angle BDA = \angle BCE$, Prop. 56.

$\therefore$ $\triangle$s ABD, BCE, are equiangular,

$\therefore BC : CE :: BD : AD$; Prop. 73.

$\therefore CE . BD = AD . BC$. Prop. 65.

Again $\because \begin{cases} \angle ABE = \angle DBC \\ \angle BAE = \angle BDC \end{cases}$ Constr. Prop. 56.

$\therefore$ $\triangle$s ABE, BDC, are equiangular:

$\therefore AB : AE :: BD : CD$ Prop. 74.

$\therefore AE . BD = AB . CD$; Prop. 65.

but, $CE . BD = AD . BC$

$\therefore AE . BD + CE . BD = AB . CD + AD . BC$

but $AE . BD + CE . BD = AC . BD$ Prop. 41.

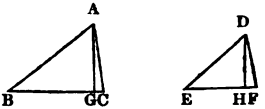
$\therefore AC . BD = AB . CD + AD . BC$.

Therefore the rectangle, &c.

PROP. LXXXI. THEOR.

Equiangular triangles have their bases in the same ratio as their altitudes, or perpendiculars upon the bases from the opposite and equal angles.

Let the $\triangle$s ABC, DEF have the $\angle$s A, B, C of the one $\triangle$, respectively equal to the $\angle$s D, E, F of the other; draw $AG \perp BC$ *and* $DH \perp EF$; then shall $DH : AG :: EF :$ *BC*; or $DH : EF :: AG : BC$.



Hyp. Const. Since $\begin{cases} \angle ABC = \angle DEF, \\ \angle AGB = \angle DHE, \end{cases}$

$\therefore$ △s ABG, DEH are equiangular;

Prop. 73. .. $\therefore \begin{cases} \dfrac{DH}{AG} = \dfrac{DE}{AB}, \\ \dfrac{EF}{BC} = \dfrac{DE}{AB}; \end{cases}$

$\therefore \dfrac{DH}{AG} = \dfrac{EF}{BC};$

That is, DH : AG :: EF : BC;

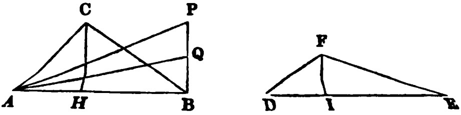
Prop. 67. or DH : EF :: AG : BC.

Therefore, equiangular △s, &c.

PROP. LXXXII. THEOR.

Triangles having equal bases are to each other as their altitudes.

Let ACB, DFE be 2 △s, having base AB = base DE. Draw CH ⊥ AB, and FI ⊥ DF, then shall △ACB : △DFE :: CH : FI.



Draw PB ⊥ AB;

$$\text{make}\begin{cases}PB = CH,\\ QB = FI;\end{cases}\qquad \text{Join}\begin{cases}AP\\ AQ\end{cases}$$

$$\therefore\begin{cases}\triangle APB = \begin{cases}\triangle ACB, \text{—having the same}\\ \quad\text{bases and altitudes,}\end{cases}\\ \triangle AQB = \begin{cases}\triangle DFE, \text{ – having equal}\\ \quad\text{bases and altitudes.}\end{cases}\end{cases}$$

Prop. 34. Cor.

Prop. 35. Cor. 1.

And $\triangle APB : \triangle AQB :: PB : QB$, Prop. 71.

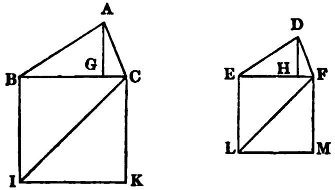
$\therefore \triangle\ ACB : \triangle DFE :: CH : FI.$

Wherefore triangles, &c.

PROP. LXXXIII. THEOR. 19. 6 Eu.

Equiangular triangles are to each other as the squares of their corresponding sides.

Let △s ABC, DEF have the ∠s A, B, C, = ∠s D, E, F respectively; then shall $\triangle\ ABC : \triangle\ DEF :: BC^2 : EF^2$.



On BC and EF draw the squares BK, EM, and the diams. CI, FL; Prop. 38.

Also draw AG ⊥ BC and DH ⊥ EF.

Since △s on = bases are as their altitudes, Prop. 82.

$$\therefore\begin{cases}\dfrac{\triangle ABC}{\triangle BIC} = \dfrac{AG}{BI} = \dfrac{AG}{BC},\\[2ex] \dfrac{\triangle DEF}{\triangle ELF} = \dfrac{DH}{EL} = \dfrac{DH}{EF},\end{cases}$$

Hyp. Also $\because \triangle$s ABC, DEF are equiangular,

Prop. 81. $$\therefore \frac{AG}{BC} = \frac{DH}{EF};$$

Ax. 1. $$\therefore \frac{\triangle ABC}{\triangle BIC} = \frac{\triangle DEF}{\triangle ELF};$$

Ax. 7. $$\therefore \frac{\triangle ABC}{2\triangle BIC} = \frac{\triangle DEF}{2\triangle ELF};$$

$$\text{but} \begin{cases} 2\triangle BIC = \text{Sq. } BK = BC^2 \\ 2\triangle ELF = \text{Sq. } EM = EF^2 \end{cases}$$

$$\therefore \frac{\triangle ABC}{BC^2} = \frac{\triangle DEF}{EF^2}$$

That is $\triangle ABC : BC^2 :: \triangle DEF : EF^2$,

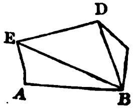
Prop. 67. or $\triangle ABC : \triangle DEF :: BC^2 : EF^2$.

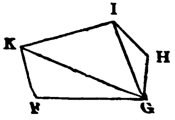
Therefore, equiangular $\triangle$s, &c.

PROP. LXXXIV. THEOR.

Similar rectilinear figures are to each other as the squares of their like or corresponding sides.

Let ABCDE, FGHIK be any two similar figures, having the $\angle$s A, B, C, &c., of the one figure respectively equal to the $\angle$s F, G, H, &c., of the other, and the sides about the equal $\angle$s proportional; also let AB and FG be corresponding sides; then shall fig. ABCDE : fig. FGHIK :: $AB^2 : FG^2$.





Draw BE. BD, GK, GI dividing the figs. into an = number of △s, by lines from the equal ∠s B and G.

Then ∵	∠A = ∠F	Hyp.
	AB : AE :: FG : FK,	..
∴	△s ABE, FGK are equiangular,	Prop. 74.
	∠AEB = ∠FKG;	..
Also ∵	∠C = ∠H,	Hyp.
	BC : CD :: GH : HI,	..
∴	△s BCD, GHI are equiangular.	Prop. 74.
	∠CDB = HIG.	..
Again ∵	∠AED = ∠FKI,	Hyp.
	∠AEB = ∠FKG,	
∴ rem. ∠BED = rem. ∠GKI.		
and ∵	∠CDE = ∠HIK,	Hyp.
	∠CDB = ∠HIG,	
∴ rem. ∠BDE = rem. ∠GIK;		
∴ △s BDE, GIK are equiangular.		

∴ each △ of the one fig. is equiangular and similar to each corresponding △ of the other fig.

Also since	$\frac{AB}{FG}=\frac{BC}{GH}=\frac{DE}{IK}$,	Hyp.
∴	$\frac{AB^2}{FG^2}=\frac{BC^2}{GH^2}=\frac{DE}{IK^2}$;	Prop. 69.
But	$\frac{AB^2}{FG^2}=\frac{\triangle ABE}{\triangle FGK}$,	Prop. 83.
	$\frac{BC^2}{GH^2}=\frac{\triangle BCD}{\triangle GHI}$,	..
	$\frac{DE^2}{IK^2}=\frac{\triangle BDE}{\triangle GIK}$;	..

Ax. 1. $\therefore \frac{AB^2}{FG^2} = \frac{\triangle ABE}{\triangle FGK} = \frac{\triangle BCD}{\triangle GHI} = \frac{\triangle BDE}{\triangle GIK}$;

Prop. 68. $\therefore AB^2 : FG^2 :: \triangle ABE + \triangle BCD + \triangle BDE : \triangle FGK + \triangle GHI + \triangle GIK$;

or $AB^2 : FG^2 ::$ whole fig. ABCDE : whole fig. FGHIK.

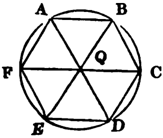
In like manner it may be proved that similar rectilinear figures of any number of sides, are as the squares of their like or corresponding sides.

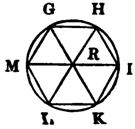
Wherefore similar rectilinear figures, &c.

PROP. LXXXV. THEOR.

The circumferences of equilateral polygons, which have the same number of sides, have the same ratio as the radii of their circumscribing circles.

Let ABCDEF, GHKILM be 2 equilat. polygons having an = no. sides inscrib. in $\odot$s whose centres are Q, R; join AQ, BQ, &c. and GR, HR, &c.: then shall $\odot^{ce}$ of polygon ABC, &c. :: $\odot^{ce}$ of polygon GHI &c. :: AQ : GR.





15 def.
yp.

Since $\begin{cases} FQ, AQ = AQ, BQ \text{ ea. to ea.,} \\ \text{and } FA = AB, \end{cases}$

∠ AQF = ∠ AQB; Prop. 7.
Similarly ∠ AQB = ∠ BQC, &c.;
∴ ∠s at cent. Q. subtended by each side are equal to each other.
Similarly, ∠s at cent. R subtended by each side = each other.
∴ ∠ AQF = ∠ GRM being like parts of 4 rt. ∠s. Prop. 70.
Also ∵ the △s in each polygon are isosceles,
∴ ∠s QAF, QFA = ∠s RGM, RMG ea. to ea. Prop. 5.
∴ △s AQF, GRM are equiangular.

$$\therefore \frac{AQ}{GR} = \frac{AB}{GH} = \frac{BC}{HI} = \frac{CD}{IK} \text{ \&c.}$$ Prop. 73.

∴ AQ : GR :: AB + BC + CD, &c. : GH + HI + IK, &c. Prop. 68.
i.e. AQ : GR :: $\odot^{ce}$ of polygon ABC, &c. : $\odot^{ce}$ of polygon GHI, &c.

Therefore, the $\odot^{ces}$ &c.

Cor.—If the sides be indefinitely diminished, and their no. increased, the $\odot^{ces}$ of the polygons will coincide with the $\odot^{ces}$ of the circumscribing ⊙s ∴ the circumferences of circles are to other as each their radii, or as their doubles, viz., their diameters. Prop. 70.

PROP. LXXXVI. THEOR.

The areas of circles are to each other as the *squares of their radii*, or as the squares of *their diameters.* (*Figs.* last prop.)

H

).83. For △ AQF : △ GRM : : AQ^2 : GR^2;
and equimultiples of △s AQF : GRM ha
).70. the same ratio;
∴ Area of polygon ABC, &c. : area of pol
gon GHI, &c. : : AQ^2 : GR^2.
If the sides be indefinitely diminished, an
their number increased, the areas of the tw
polygons will be equal to the areas of their ci
cumscribing ⊙s;
∴ Area of ⊙ ABC, &c. : area of ⊙ GHI &
: : AQ^2 : GR^2,
).70. or : : AD^2 : GK^2.
Therefore the areas of circles, &c.

EXERCISES.

1. In a given straight line, to find a poi equally distant from two given points.

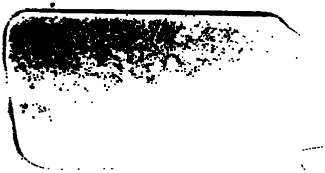
2. Given one side of a right-angled triangle, and the difference between the hypothenuse and the other side; show how the triangle can be determined.

3. The straight line which bisects the angle opposite the base of an isosceles triangle, bi sects the base also, and is at right angles to i

4. In a given straight line to find a poi *from* which, if lines be drawn to two giv *points on the* same side of the line, these li *may make* equal angles with it.

5. *If the* base of an iso

[illegible]



15. If an angle of a triangle and its opposite side be bisected by a straight line; the line is perpendicular to the side, and the triangle is isosceles.

16. Trisect a given straight line.

17. Trisect a right angle.

18. The greatest diameter of a parallelogram is that which is opposite to the greatest angle.

19. If the opposite sides of a quadrilateral figure are equal, it is a parallelogram.

20. If the opposite angles of a quadrilateral figure are equal, it is a parallelogram.

21. The difference between the angles at the base of a triangle is equal to twice the angle made by the straight line which bisects the remaining angle and the perpendicular from its vertex to the base.

22. From a given point in a side of a parallelogram, to draw a straight line that will bisect the parallelogram.

23. From a given point in the side of a triangle, to draw a straight line that will bisect the triangle.

24. Equal triangles between the same parallels, are upon equal bases.

25. If a straight line which is parallel to either side of a triangle bisect one of the other sides, it will also bisect the remaining side.

26. In figure to Prop. 39, if FD, GH, KE, be joined, the triangles FBD, GAH, KCE, are each equal to the triangle ABC.

27. If the square described upon one of the sides of a triangle be equal to the sum of the

; described upon the other two sides of angle contained by these two sides is angle.

One side of a right-angled triangle is ıd the other side 367; find the hypo- : by means of Prop. 39.

One side of a right-angled triangle is ıd the hypothenuse 824; find the re- ȝ side by the same Prop.

If the sides of a right-angled triangle be ', and the hypothenuse h, show alge- y that $a=\overline{(h+b)\,(h-b)}^{\frac{1}{2}}$.

The sum of the 4 lines drawn from a vithin a trapezium to the 4 angles, is st, when that point is the intersection liameters.

If one angle of a triangle be equal to er two together, it is a right angle.

The diameters of a rhombus bisect each t right angles.

The diameters of a parallelogram bisect her.

If two exterior angles of a triangle be l, and from the point of intersection of the ıg lines, a line be drawn to the opposite ıf the triangle, it will bisect that angle.

Inscribe a square in a given right- isosceles triangle.

Inscribe a square in a given quadrant 'cle.

Inscribe a circle in a given quadrant :le.

f two straight lines cut each other at

right angles, and their extremities be joined, the sums of the squares of the opposite sides of the trapezium thus formed, are equal to each other.

40. The diameter is the greatest straight line that can be drawn in a circle.

41. If a perpendicular be drawn to a side of a triangle from the opposite angle; the rectangle of the sum and difference of the segments of that side, is equal to the rectangle of the sum and difference of the other sides.

42. The sides of a triangle are 3, 5, and 7; determine the numerical values of the perpendicular upon the longest side from the opposite angle, and also the segments of that side.

43. In any triangle, if a line from the middle of one side to the opposite angle be drawn; double the sum of the squares of the line, and of half that side is equal to the sum of the squares of the other two sides.

44. If two points be taken in the diameter of a circle equally distant from the centre; the sum of the squares of two lines drawn from these points to any point in the circumference will always be the same.

45. The sum of the squares of the two diameters of a parallelogram, is equal to the sum of the squares of the four sides.

46. The square of either side of an equilateral triangle is equal to three times the square of the radius of the circumscribing *circle.*

47. If two circles cut each other, they can*not have the same* centre.

48. Perpendiculars from the centre of a circle to equal straight lines within the circle, are equal; and if the perpendiculars are equal, the lines are equal.

49. Draw a tangent to two given unequal circles.

50. Tangents to a circle from the same point, are equal to each other.

51. In equal circles, the angles which stand upon equal arcs are equal.

52. The difference between the hypothenuse and the sum of the other two sides of a right-angled triangle, is equal to the diameter of the circle inscribed in the triangle.

53. Inscribe a circle in a given triangle.

54. If an angle of a triangle be divided into two equal angles, by a straight line which cuts a side; the segments of that side will have the same ratio which the other sides of the triangle have to each other.

55. The four triangles into which a trapezium is divided by its diameters, are proportionals.

56. The triangle cut off by joining the middle points of the sides of a triangle, is one-fourth of the whole.

57. The line which joins the middle point of the hypothenuse of a right-angled triangle and the right angle, is equal to half the hypothenuse.

58. Divide a given straight line in the same *proportion* as another given divided line.

59. Find a fourth proportional to three *given straight lines*

60. Find a mean proportional between two given straight lines.

61. Bisect a triangle by a line drawn parallel to one of its sides.

62. The sum of the perpendiculars drawn from any point within an equilateral triangle to the sides, is equal to the perpendicular from either angle to its opposite side.

63. If from the extremities of any chord in a circle, perpendiculars be drawn meeting a diameter ; the points of intersection are equally distant from the centre.

64. The perpendiculars from the middle points of the sides of a triangle, will all meet in a point.

65. The perpendiculars from the angles of a triangle to the opposite sides, will all meet in a point.

66. If from the three angles of a triangle lines be drawn to the middle points of the opposite sides, they will all meet in one point.

67. The three straight lines which bisect the angles of a triangle, meet in a point.

68. If from the three angles of a triangle lines be drawn to the points of bisection of the opposite sides ; the squares of the distances between the angles and the common intersection, are together equal to one-third of the sum of the squares of the sides of the triangle.

69. If the diameter of a given circle be divided into any number of parts, on which circles are described, the circumferences of the latter will be together equal to the circumference of the given circle.

70. The circumference of a circle whose diameter is 1 is very nearly 3·1416; show that for any other diameter *d*, the circumference is $d \times 3{\cdot}1416$.

71. The area of a circle whose diameter is 1 is very nearly 0·7854; show that for any other diameter *d*, the area is $d^2 \times 0{\cdot}7854$.

THE END.

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